Fundamentals of Microlensing

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2026 Sagan Summer Workshop

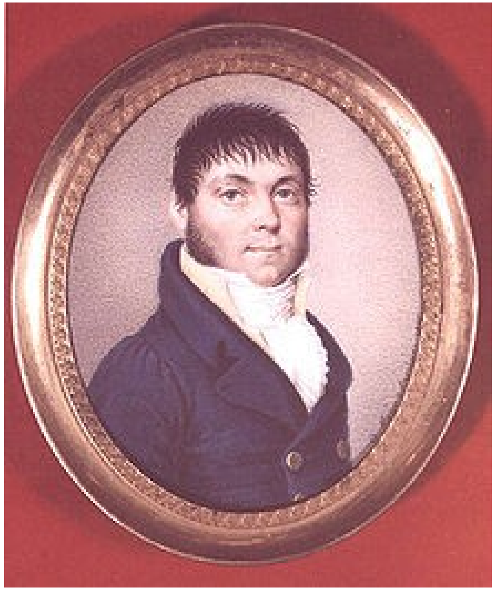
21 July 2026

UTINAM INSTITUT

UNIVERSITÉ MARIE & LOUIS PASTEUR

OSU THETA

The story starts in 1804...



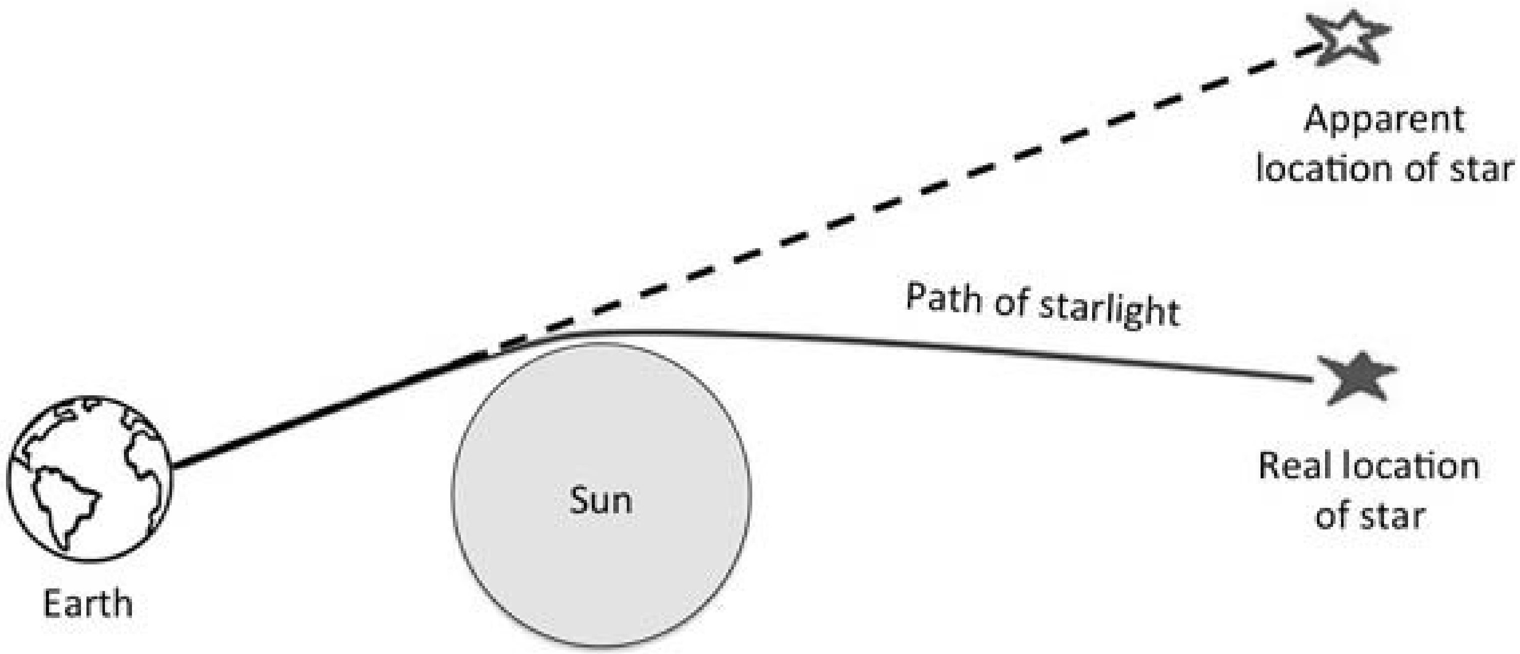
Johann Georg von Soldner
(1776-1833)

”If it were possible to observe the fixed stars very nearly at the Sun, then we would have to take this into consideration. However, as it is well known that this doesn’t happen, then also the perturbation of the Sun shall be neglected.”

On the deflection of a light ray from its rectilinear motion, by the attraction of a celestial body at which it nearly passes by

Soldner 1804

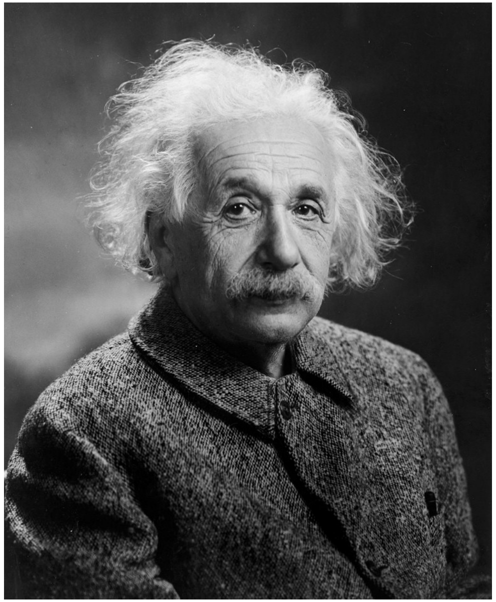
The story starts in 1804...



Prediction Soldner

$\alpha_N \sim 1''$

...in 1916 , the general relativity...



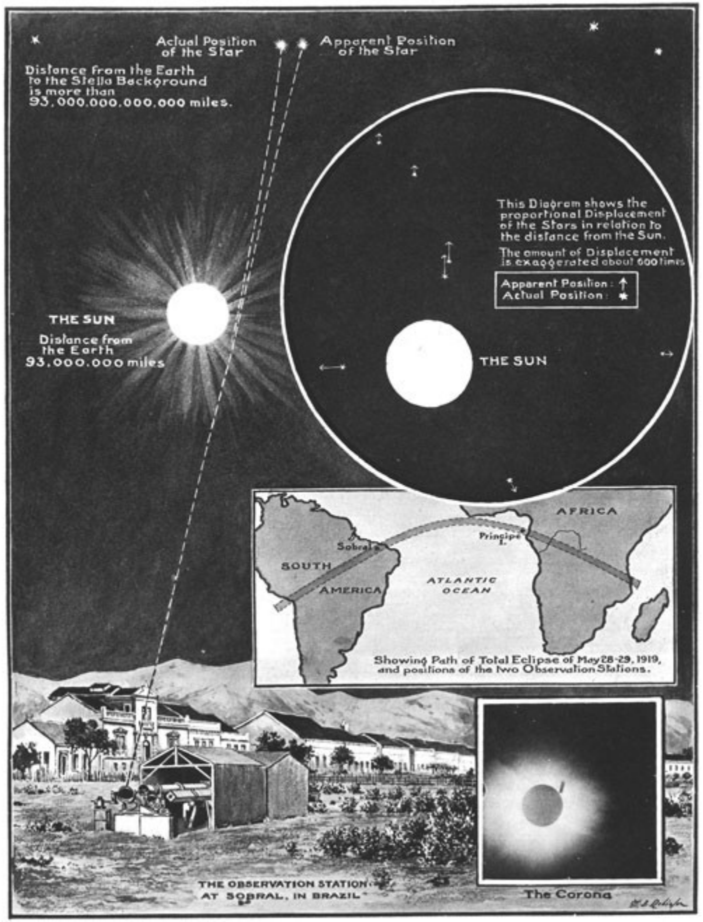
Albert Einstein
(1879-1955)

Prediction Einstein

$\alpha_E = 2\ \alpha_N \sim 2''$

...in 1919 , the Eddington experiment

Arthur Eddington
(1882-1944)



...in 1919 , the Eddington experiment

Arthur Eddington
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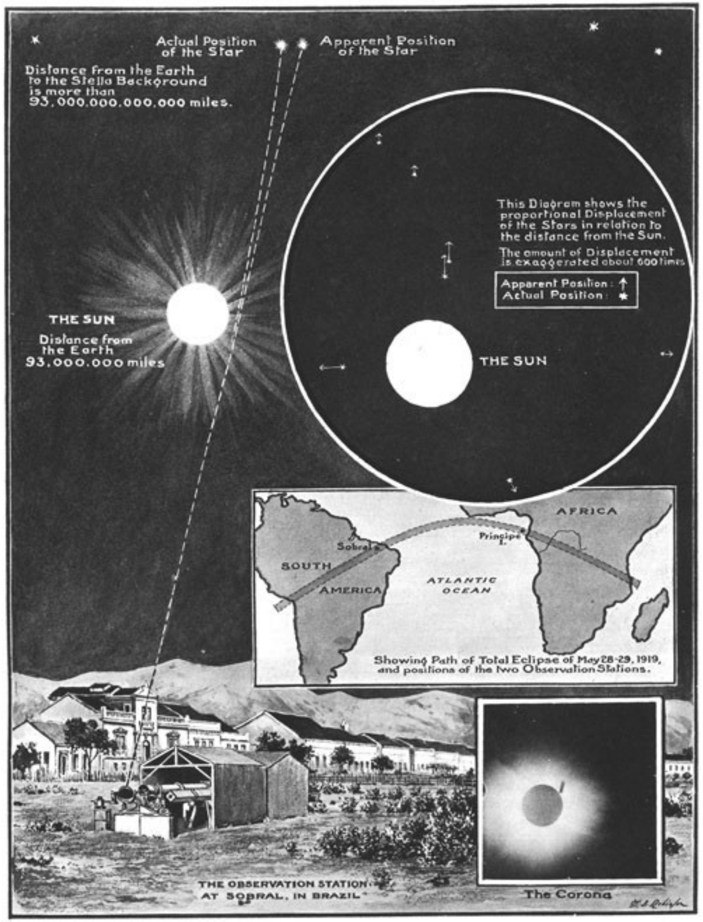
Eddington measures

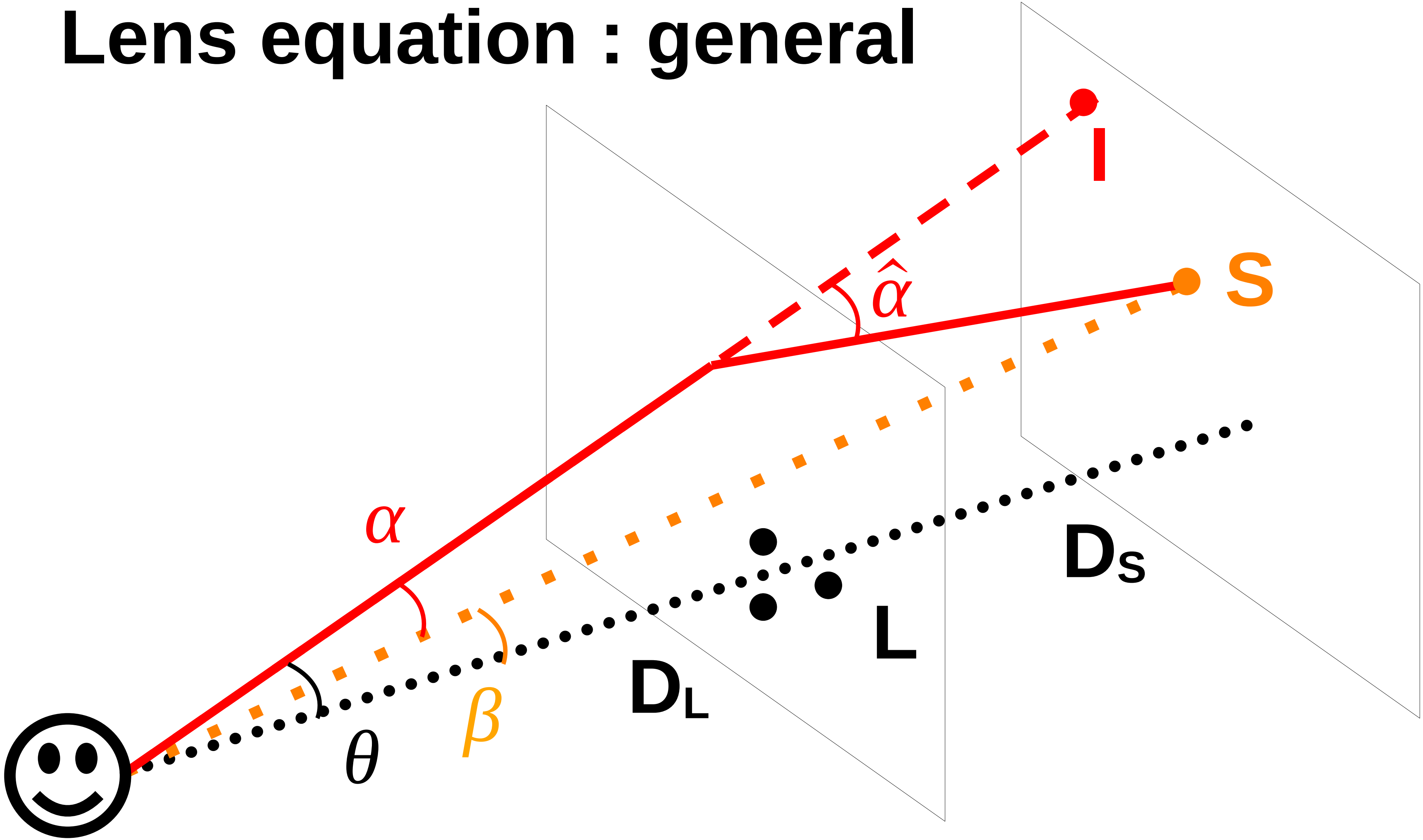
$\alpha_M = 2'' = \alpha_E$



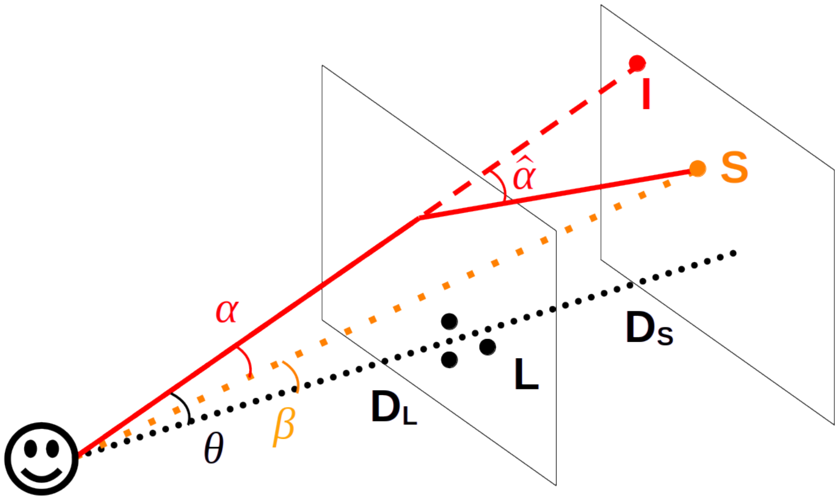
Lens equation : hypothesis

- **Point mass lens** ~~odd number theorem~~
- **Geometric optic limit** $R_E \gg \lambda$
- **Thin lens approximation** $R_L \ll D_L$
- **Weak deflection** $\alpha \ll 1$

Lens equation : general



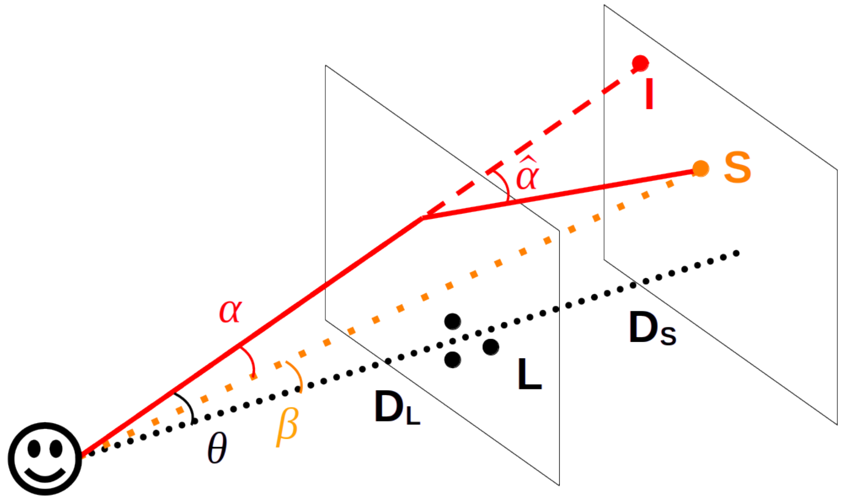
Lens equation : general



Lens equation

$$\theta = \beta + \alpha$$

Lens equation : general



Lens equation

$$\boldsymbol{\theta} = \boldsymbol{\beta} + \boldsymbol{\alpha}$$

Apparent deflection angle (General Relativity)

$$\pi_{rel} = \frac{1}{D_L} - \frac{1}{D_S}$$

$$\boldsymbol{\alpha} = \frac{4G\pi_{rel}}{c^2} \sum_{i=1}^{N} m_i \frac{\boldsymbol{\theta} - \boldsymbol{\theta_i}}{|\boldsymbol{\theta} - \boldsymbol{\theta_i}|^2}$$

Lens equation : general

Lens equation

$$\boxed{\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{4G\pi_{rel}}{c^2} \sum_{i=1}^{N} m_i \frac{\boldsymbol{\theta} - \boldsymbol{\theta_i}}{|\boldsymbol{\theta} - \boldsymbol{\theta_i}|^2}}$$

Lens equation : general

Lens equation

$$\boxed{\boldsymbol{\beta} = \boldsymbol{\theta} - \frac{4G\pi_{rel}}{c^2} \sum_{i=1}^{N} m_i \frac{\boldsymbol{\theta} - \boldsymbol{\theta_i}}{|\boldsymbol{\theta} - \boldsymbol{\theta_i}|^2}}$$

Towards the Bulge

Einstein ring

$$\boxed{\theta_E = \sqrt{\kappa \pi_{rel} M_L}}$$

$$\left.\begin{array}{l} D_L = 4\,kpc \\ D_S = 8\,kpc \\ M_L = 1\,M_\odot \end{array}\right\} \theta_E \approx 1\,mas$$

$$\kappa = \frac{4G}{c^2 AU} = 8.144\,mas\,.\,M_\odot^{-1} \qquad M_L = \sum_{i=1}^{N} m_i$$

Lens equation : general

Complex writing

$$\boxed{\zeta = z - \sum_{i=1}^{N} \frac{\epsilon_i}{\bar{z} - \bar{z}_i}}$$

$$\epsilon_i = \frac{m_i}{M_L}$$

Polynomial of order N_L^2+1

$$z = p(\zeta, z_i, \epsilon_i)$$

$$N_z \leq 5(N_L - 1)$$

Lens equation : 1 lens

$$\zeta = z - \frac{1}{\bar{z}}$$

Polynomial of order 2

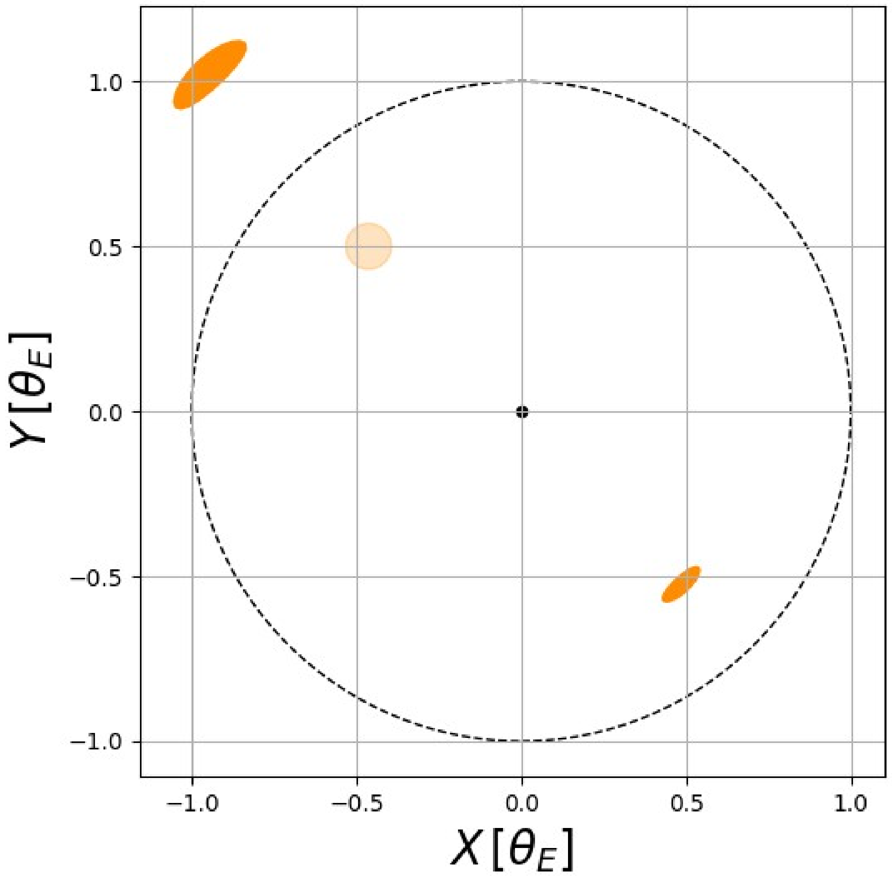
Lens equation : 1 lens

$$\zeta = z - \frac{1}{\bar{z}}$$ Polynomial of order 2

Image positions

$$z_{1,2} = \frac{\zeta \pm \sqrt{\zeta^2 + 4}}{2}$$

$$\Delta z_{1,2} = \sqrt{\zeta^2 + 4} \sim 2 \quad \zeta \rightarrow 0$$

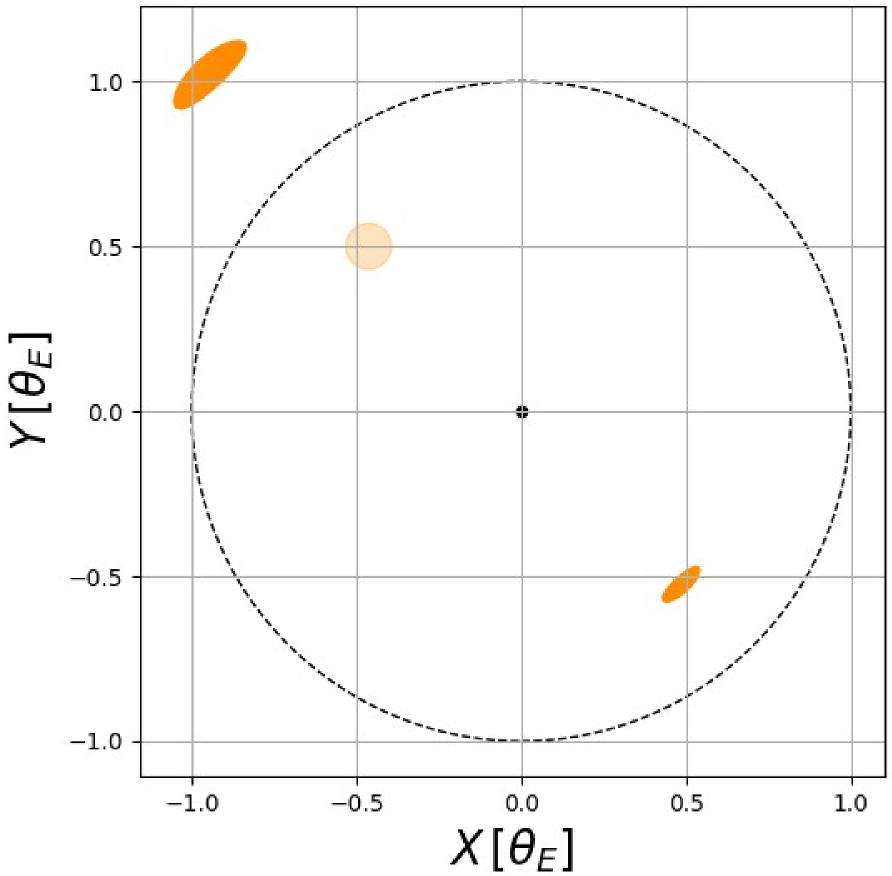


Lens equation : 1 lens

Rewriting lens equation

$$\zeta = l\, e^{i\phi}$$
$$z = r\, e^{i\omega}$$

$$\zeta = e^{i\omega}\left(r - \frac{1}{r}\right)$$



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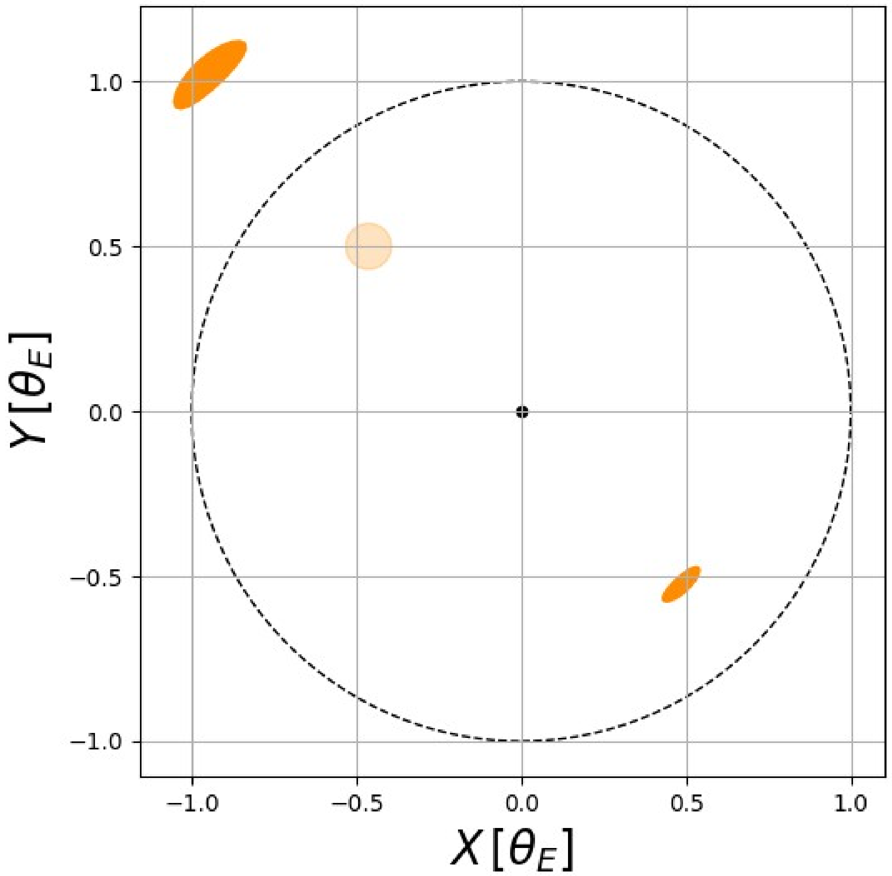
Major Image

$$\omega_1 = \phi$$

$$r_1 = \frac{l + \sqrt{l^2 + 4}}{2}$$



Lens equation : 1 lens

Rewriting lens equation

$$\zeta = l\, e^{i\phi}$$
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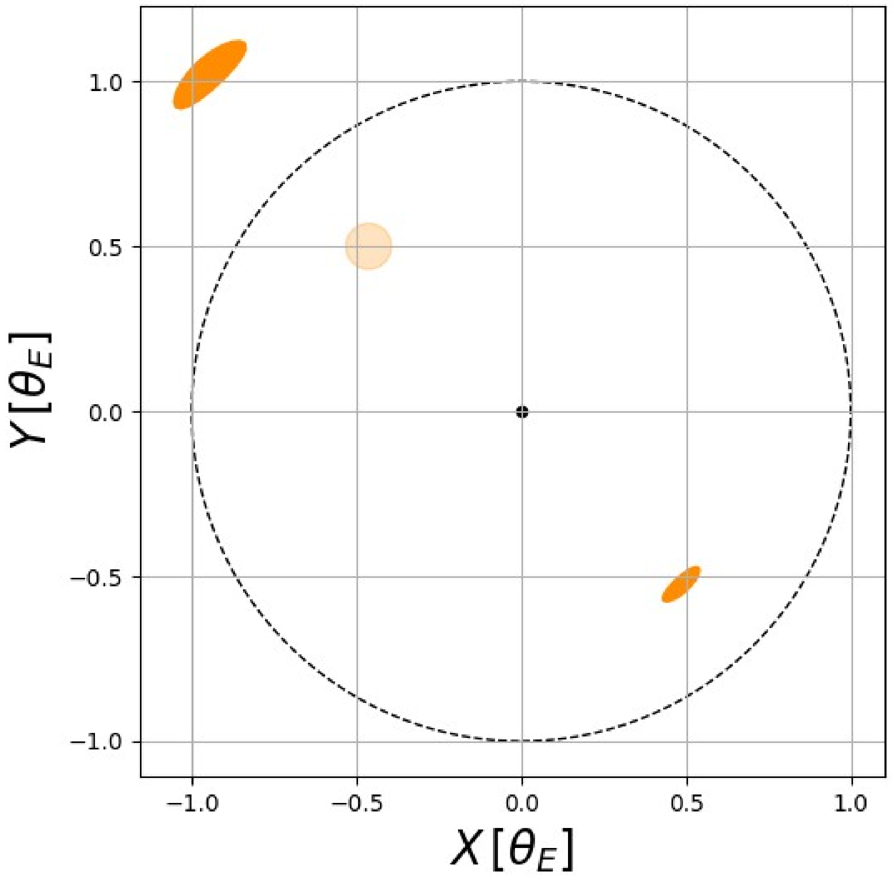
$$\zeta = e^{i\omega}\left(r - \frac{1}{r}\right)$$

Major Image

$$\omega_1 = \phi \qquad r_1 = \frac{l + \sqrt{l^2+4}}{2}$$

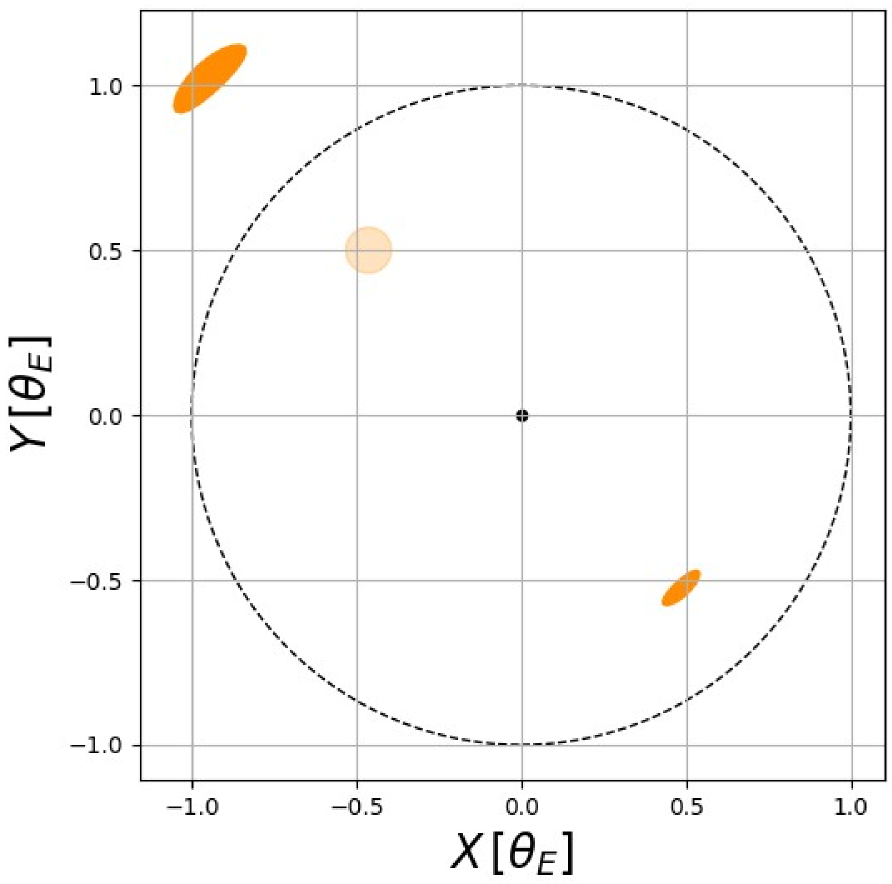
Minor Image

$$\omega_2 = \phi + \pi \qquad r_2 = \frac{\sqrt{l^2+4} - l}{2}$$



Lens equation : 1 lens

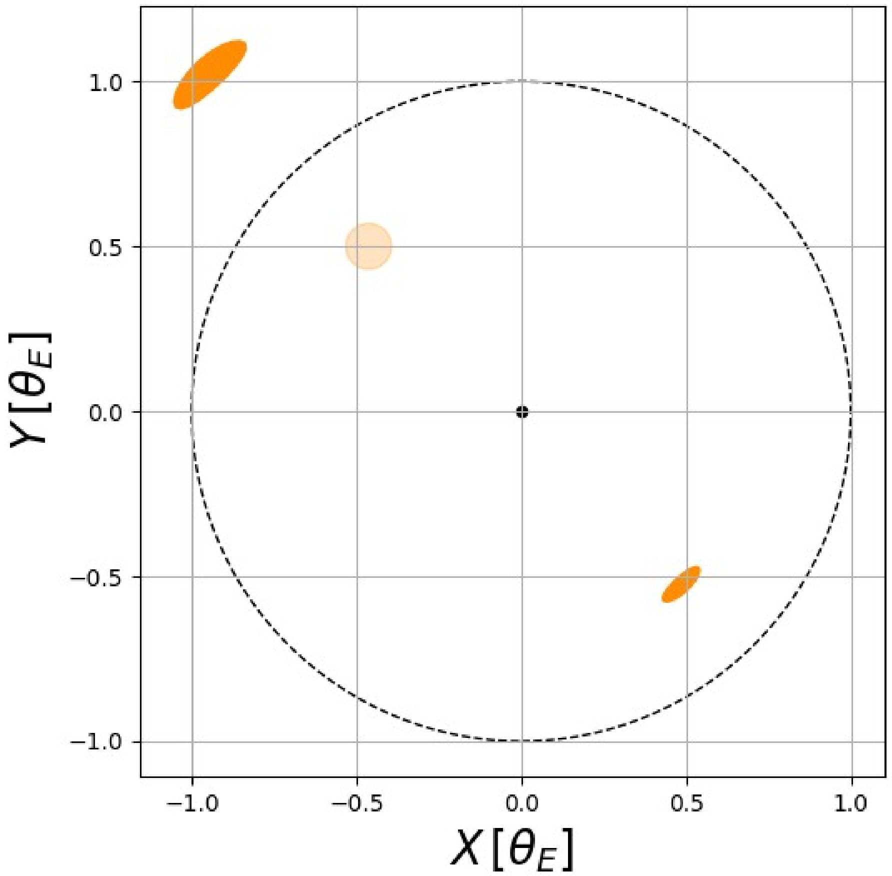
Images are deformed.
Liouville theorem states that surface brightness is conserved therefore the images are magnified !



Lens equation : 1 lens

Images are deformed.
Liouville theorem states that surface brightness is conserved, therefore the images are magnified !

$$A_{1,2} = \frac{area_{1,2}}{area_S} = \left| \frac{r\,\delta r\,\delta\omega}{l\,\delta l\,\delta\phi} \right|$$



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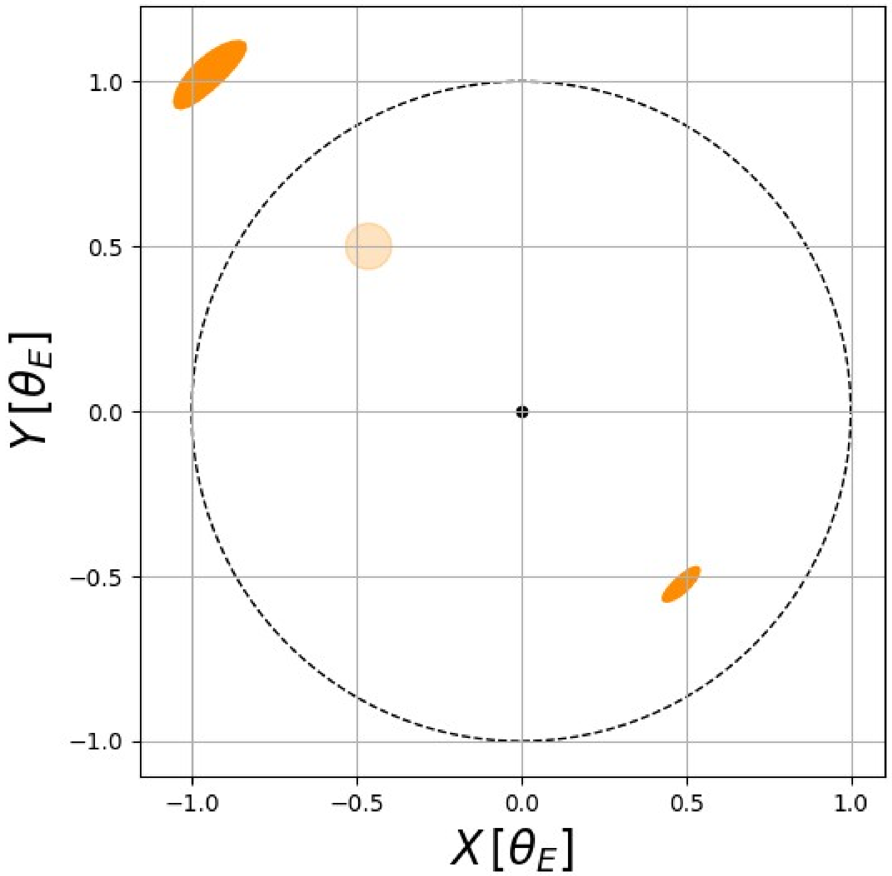
$$A_{1,2} = \frac{area_{1,2}}{area_S} = \left| \frac{r\,\delta r\,\delta\omega}{l\,\delta l\,\delta\phi} \right|$$

$$A_{1,2} = \left| \frac{1}{1 - \frac{1}{r_{1,2}^4}} \right| = \frac{1}{2}\left(\frac{\zeta^2+2}{\zeta\sqrt{\zeta^2+4}} \pm 1 \right)$$

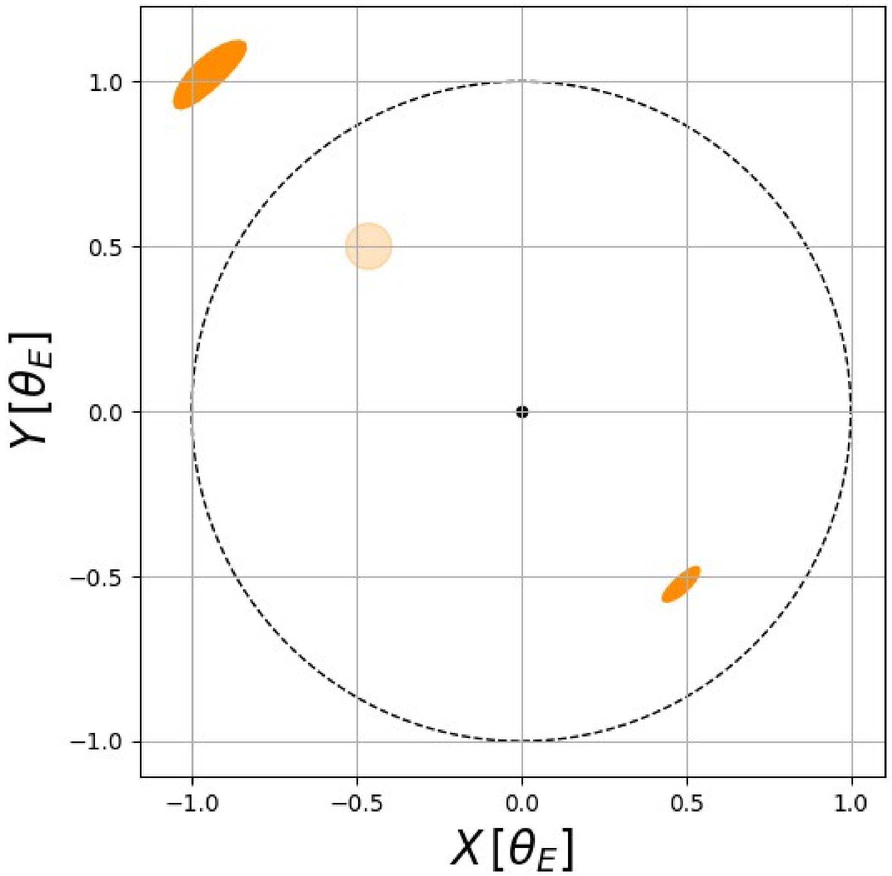


Lens equation : 1 lens

Images are not resolved,
only the total magnification is observed

$$\boxed{A = A_1 + A_2 = \frac{\zeta^2 + 2}{\zeta\sqrt{\zeta^2 + 4}}}$$

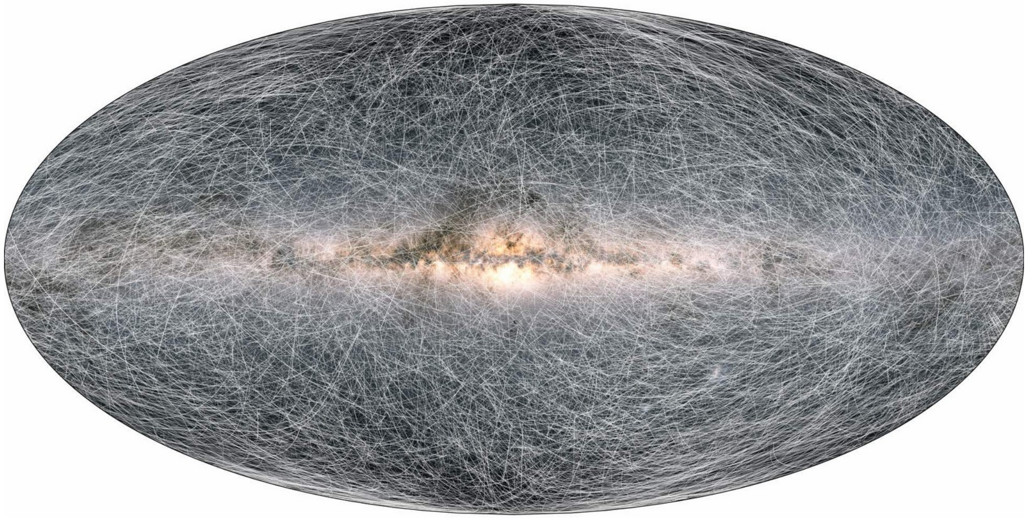
$$A = 1.55 + 0.55 = 2.10$$



Lens equation : 1 lens

But stars are moving …

$$\boldsymbol{\mu}_{rel} = \boldsymbol{\mu}_L - \boldsymbol{\mu}_S$$



Star trajectories on the sky, 400 thousands years into the future.

Credit: ESA/Gaia/DPAC

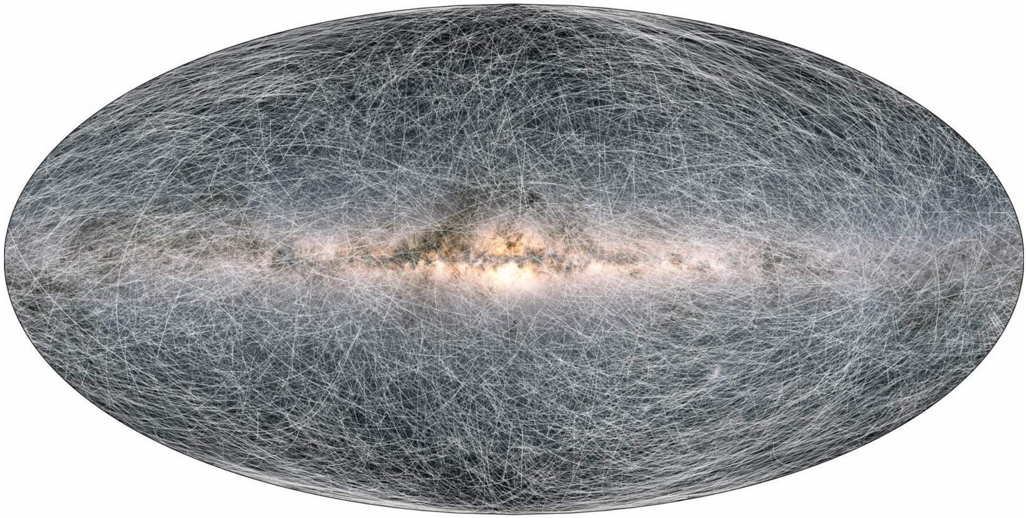
Lens equation : 1 lens

But stars are moving …

$$\boldsymbol{\mu}_{rel} = \boldsymbol{\mu}_L - \boldsymbol{\mu}_S$$

$$\boxed{t_E = \frac{\theta_E}{|\mu_{rel}|}} \quad \alpha \sqrt{M_L}$$

$$M_L = 1\, M_{Jup} \Leftrightarrow t_E \approx 1\, day$$
$$M_L = 1\, M_\odot \Leftrightarrow t_E \approx 50\, days$$
$$M_L = 10\, M_\odot \Leftrightarrow t_E \approx 200\, days$$

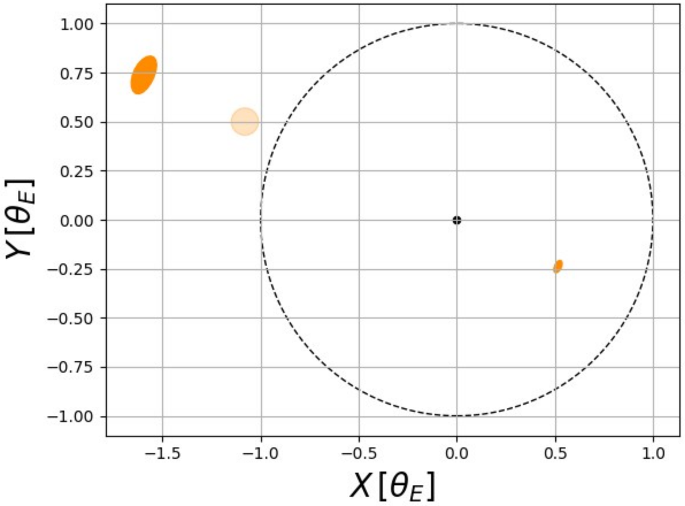


Star trajectories on the sky,
400 000 years into the future.

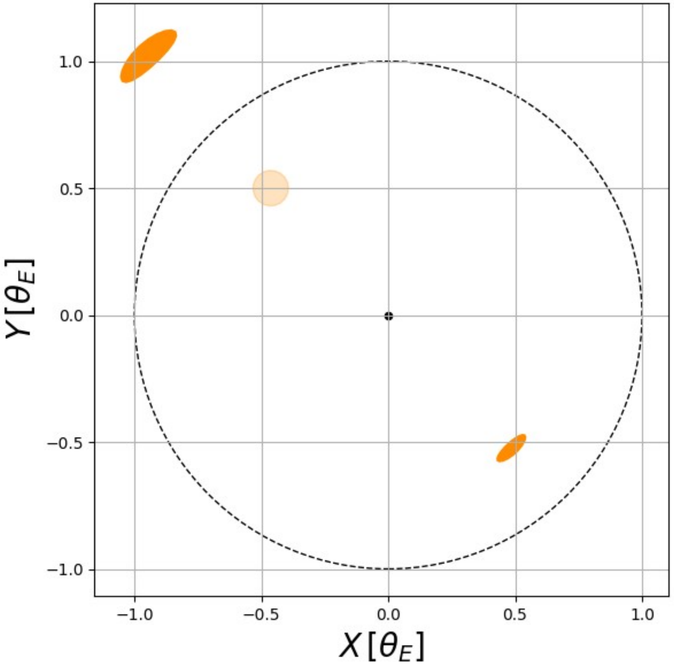
Credit: ESA/Gaia/DPAC

Lens equation : 1 lens

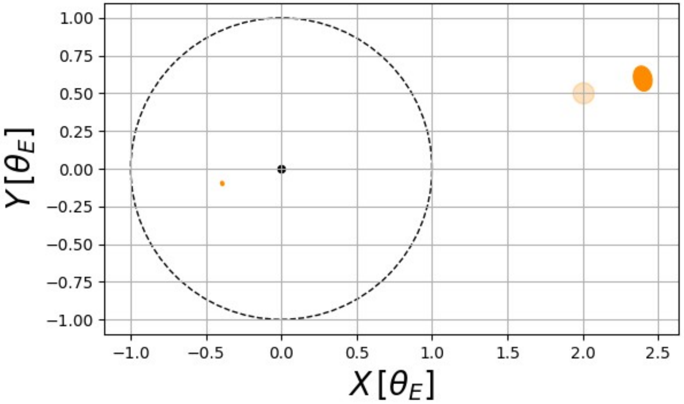
T = -35
A = 1.23



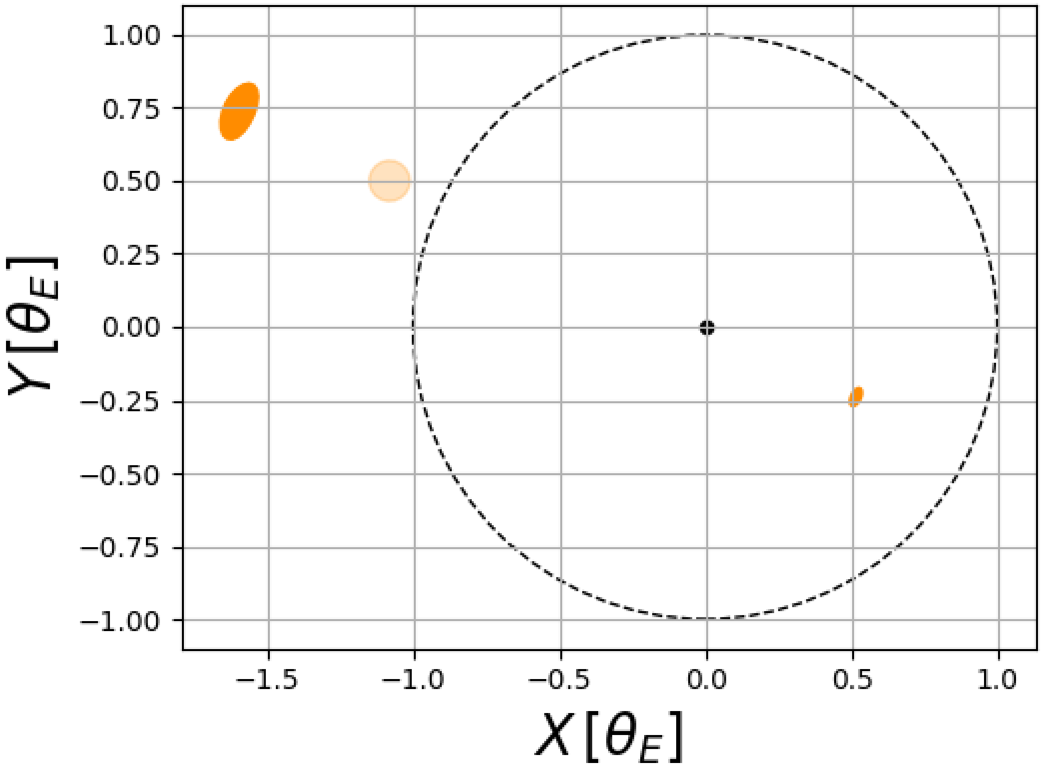
T = -15
A = 1.71



T = 65
A = 1.06



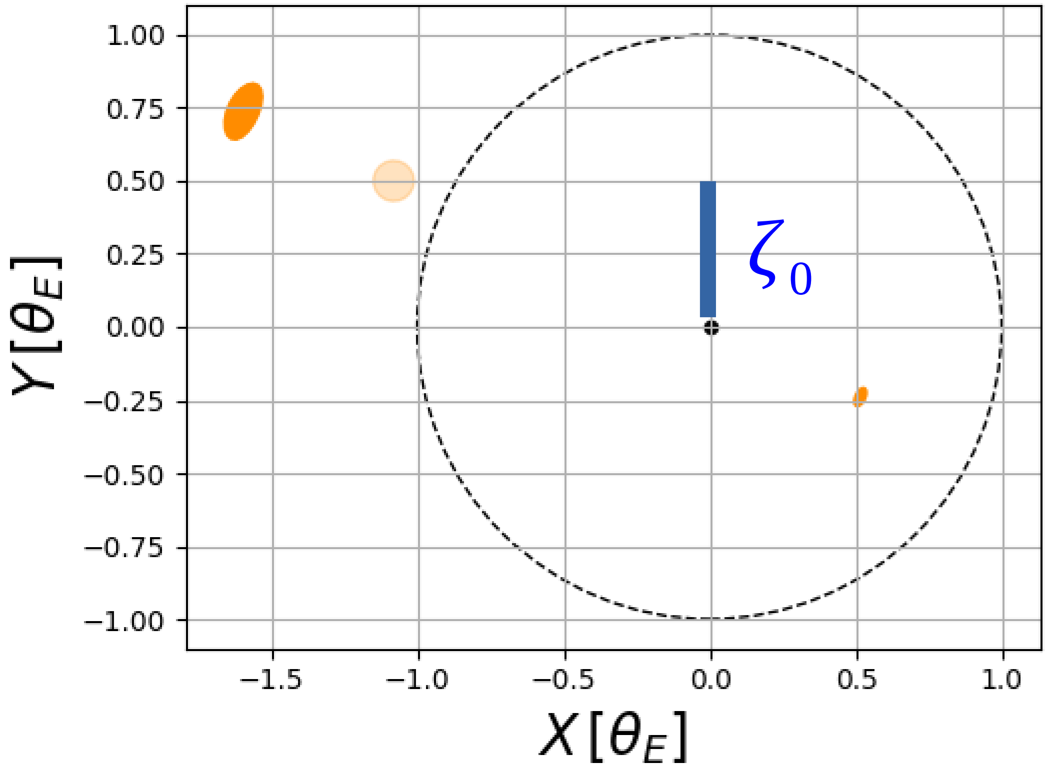
Lens equation : 1 lens



Lens equation : 1 lens

ζ_0 Minimum impact parameter

t_0 Time of minimum impact

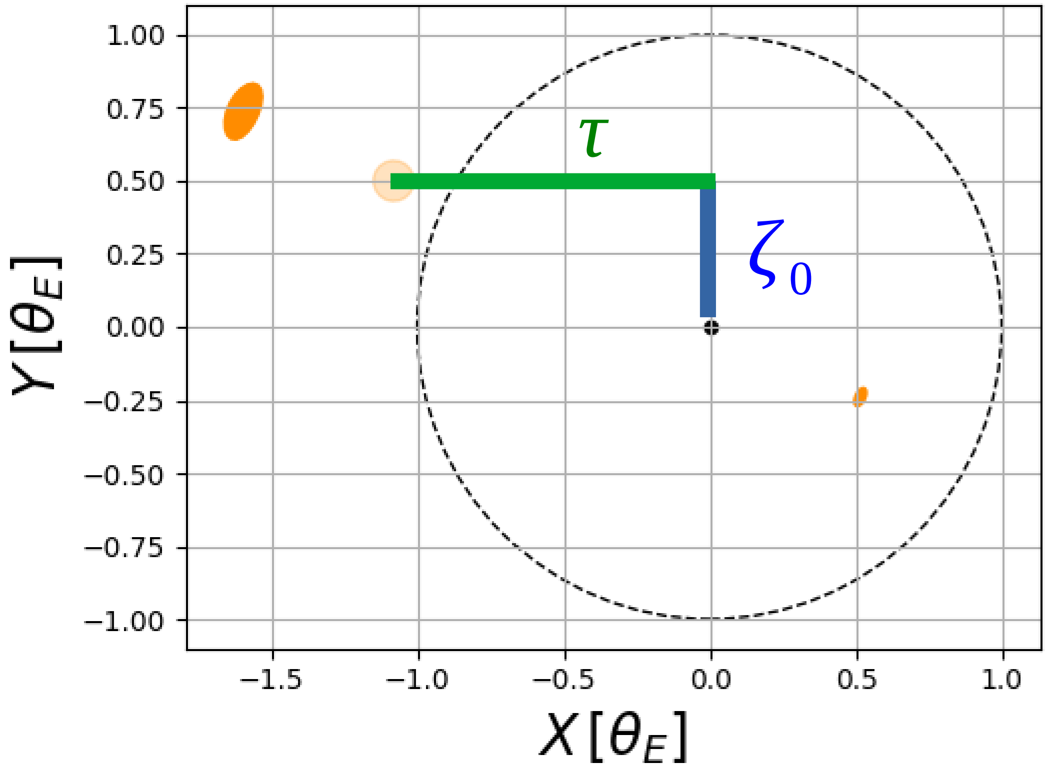


Lens equation : 1 lens

ζ_0 Minimum impact parameter

t_0 Time of minimum impact

$$\tau = \frac{(t - t_0)}{t_E}$$



Lens equation : 1 lens

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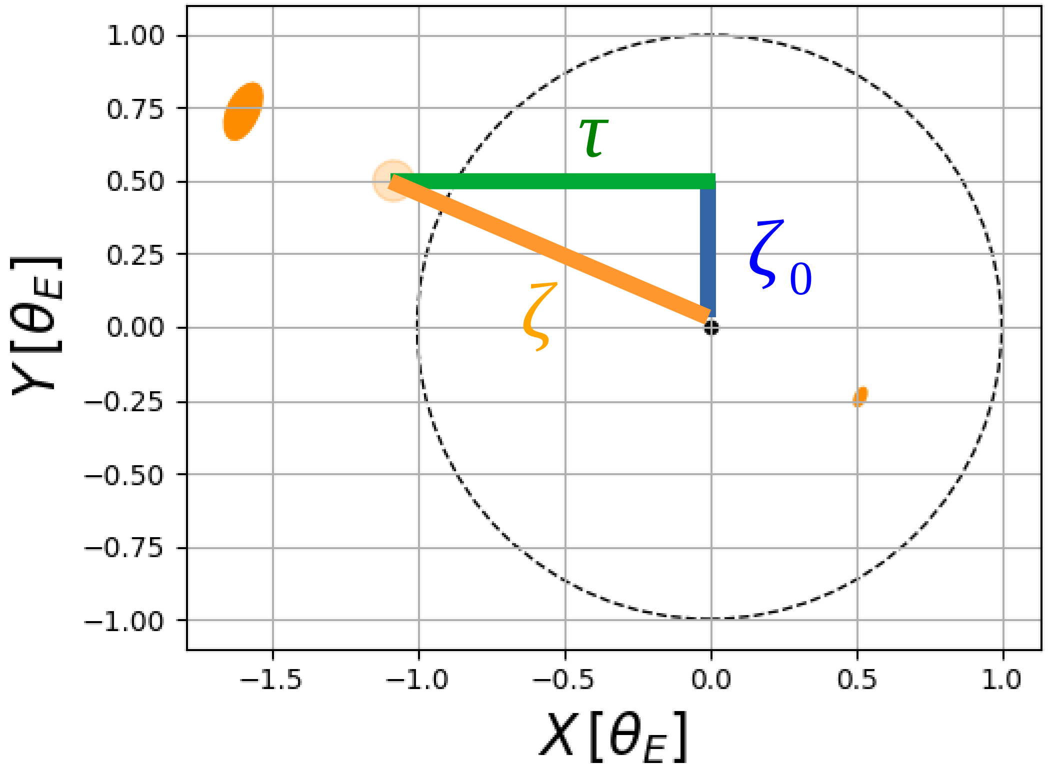
$$\tau = \frac{(t - t_0)}{t_E}$$

$$\zeta(t) = \sqrt{\tau^2 + \zeta_0^2}$$



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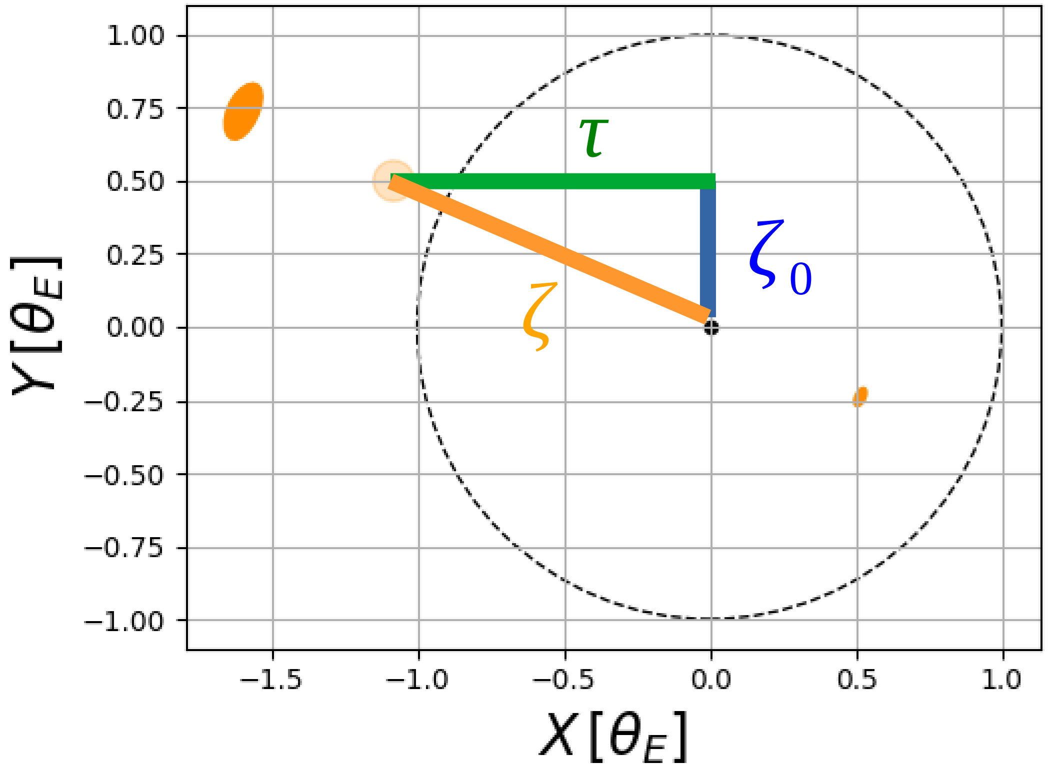
$$\zeta(t) = \sqrt{\tau^2 + \zeta_0^2}$$

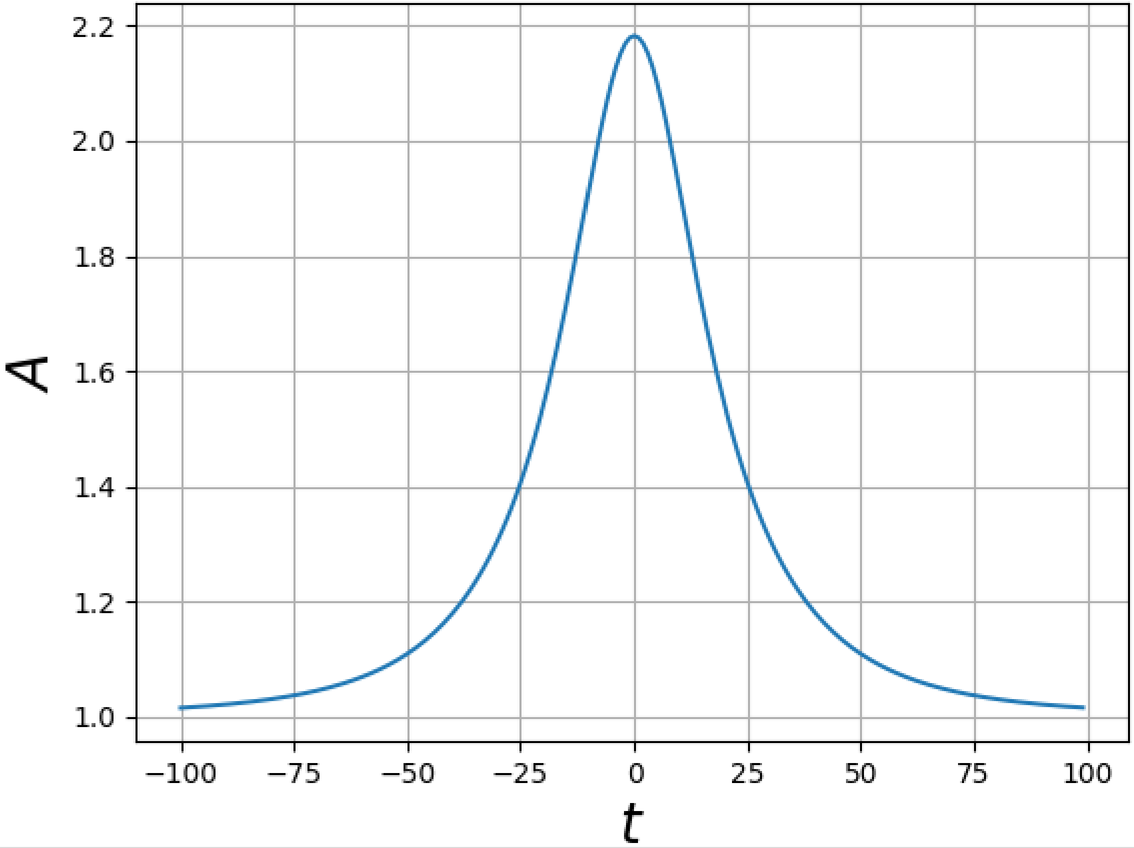
$$A(t) = \frac{\zeta(t)^2 + 2}{\zeta(t)\sqrt{\zeta(t)^2 + 4}}$$



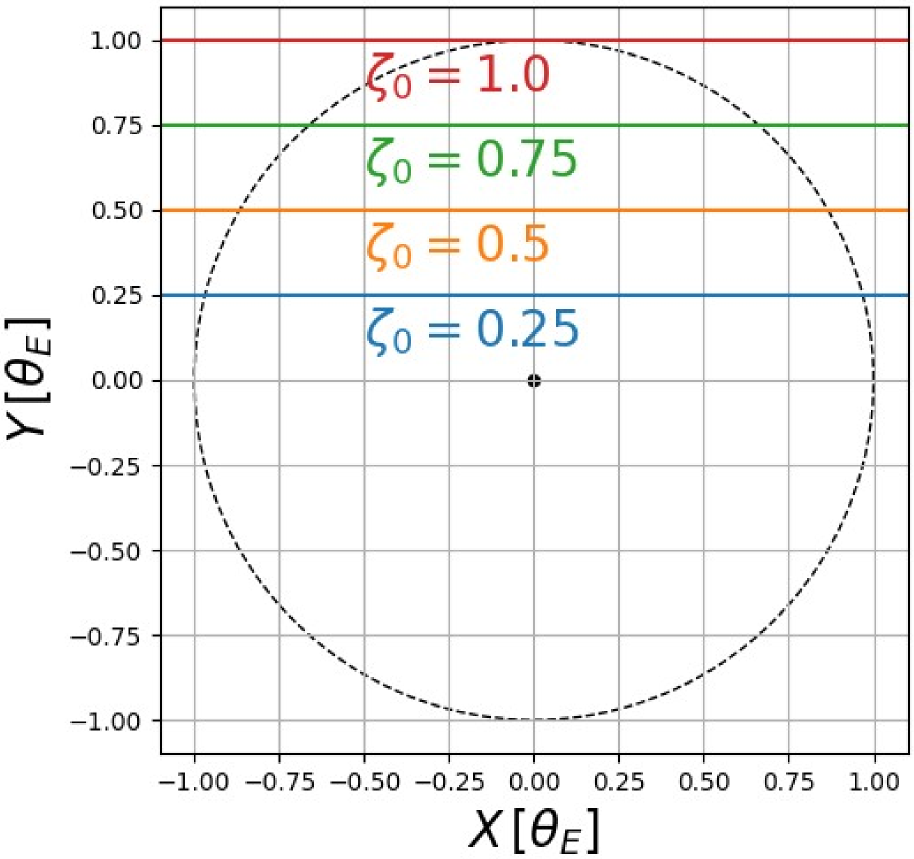
Lens equation : 1 lens

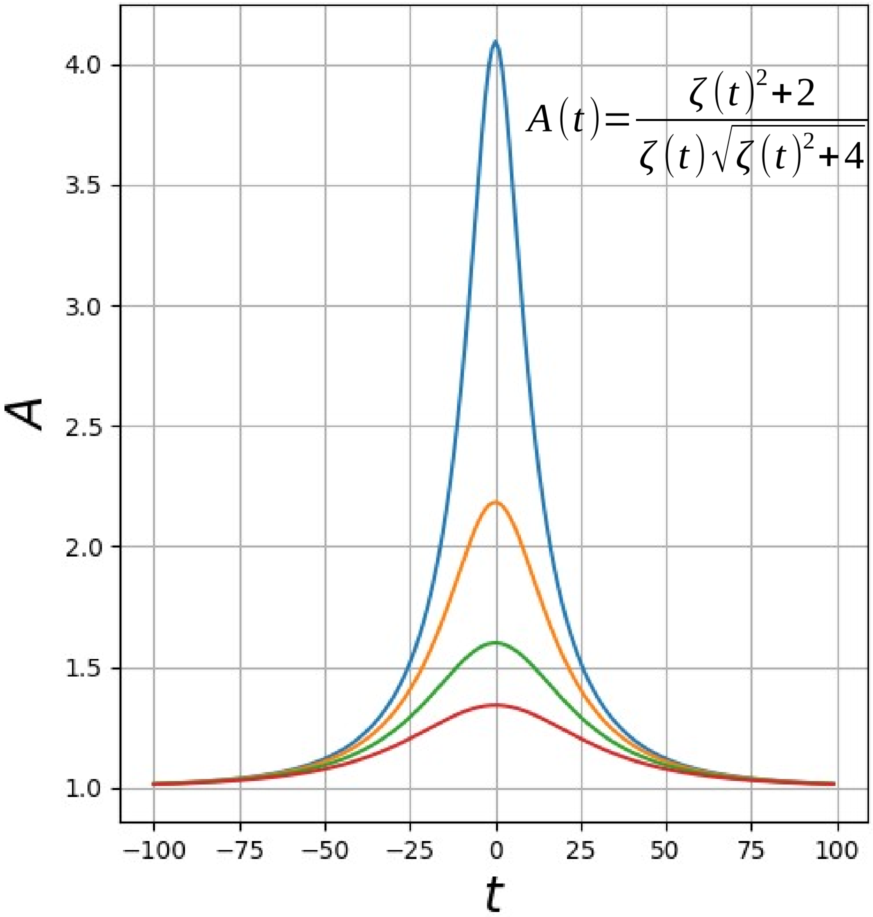
The Paczyński curve

Bohdan Paczyński
(1940 - 2007)



Lens equation : 1 lens





Lens equation : 1 lens

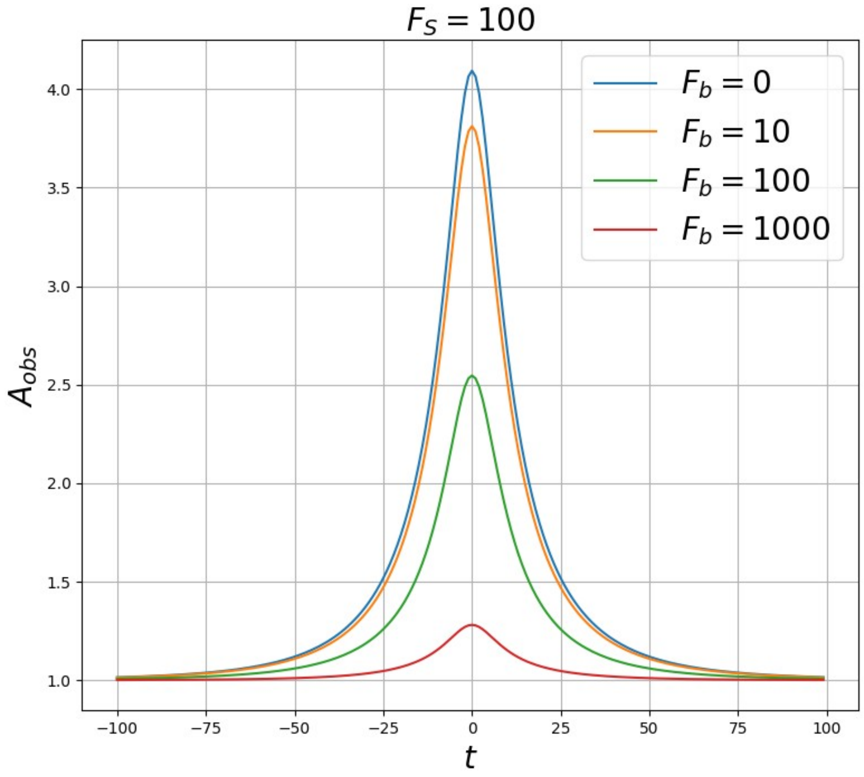
But we observe fluxes….

$F_{S,\lambda}$ Source flux at wave λ

$F_{B,\lambda}$ Blend flux at wave λ (lens, unrelated stars….)

$$\boxed{F_{tot,\lambda} = F_{S,\lambda} A(t) + F_{B,\lambda}}$$

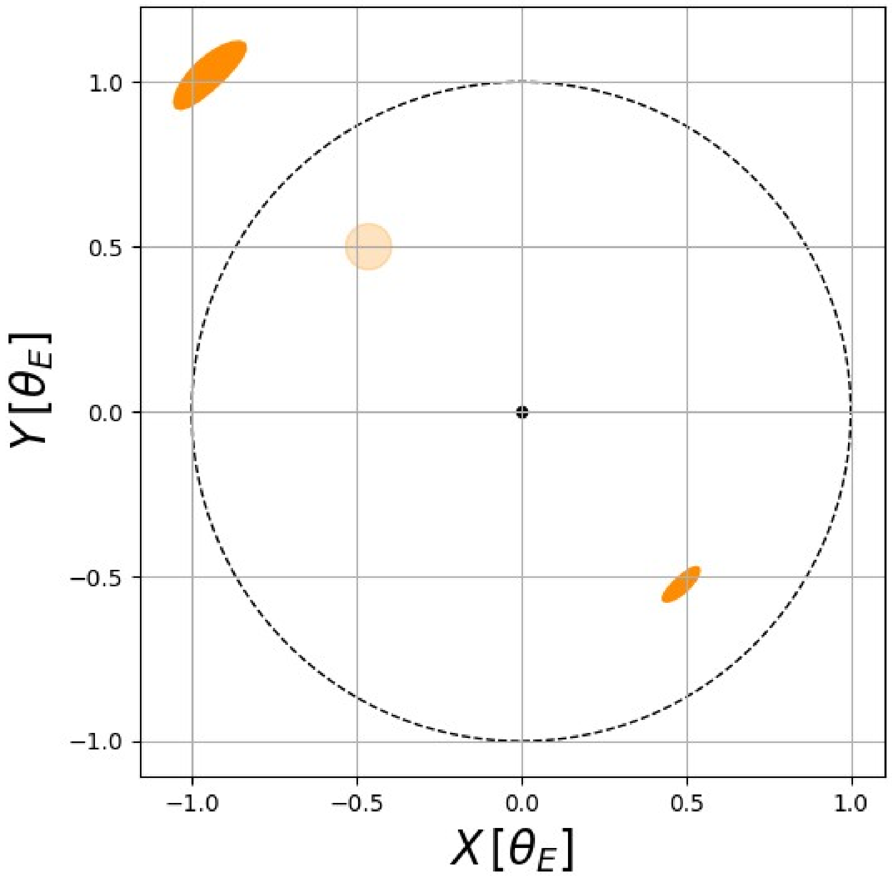
$$A_{obs}(t) = \frac{F_{tot,\lambda}}{F_{S,\lambda} + F_{B,\lambda}}$$



Lens equation : 1 lens

Images are not resolved (except with interferometry) but the light centroid moves as well!

$$\delta(t) = \frac{\sum_{i=0}^{N_I} A_i z_i}{\sum_{i=0}^{N_I} A_i} - \zeta$$



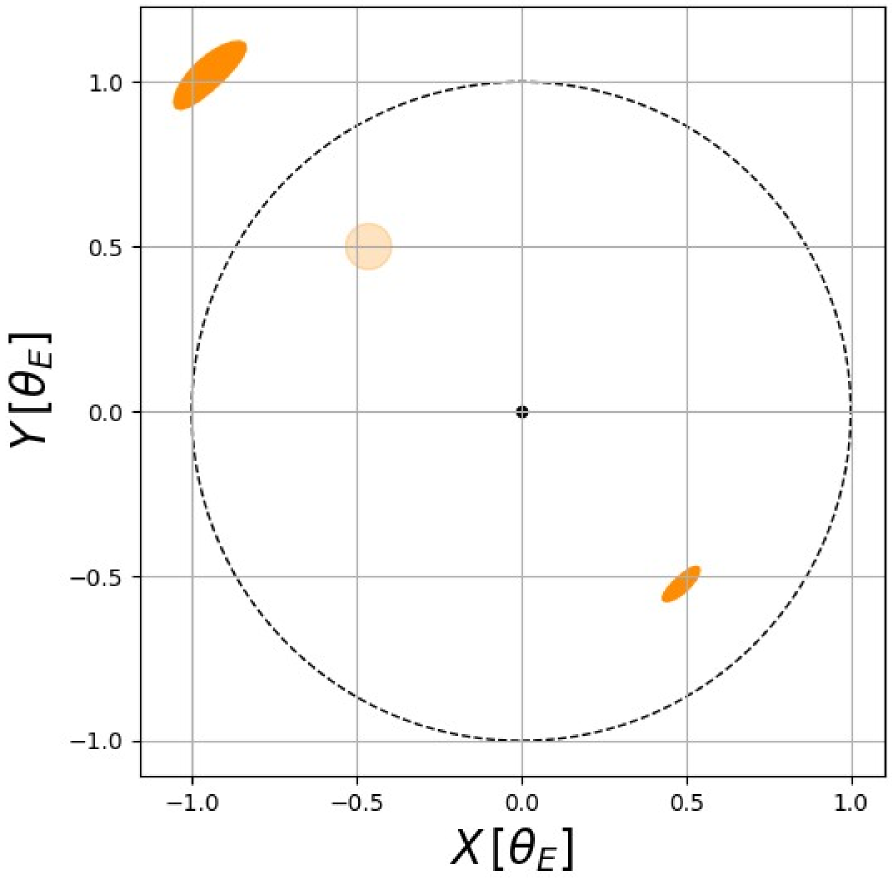
Lens equation : 1 lens

Images are not resolved (except with interferometry) but the light centroid moves as well!

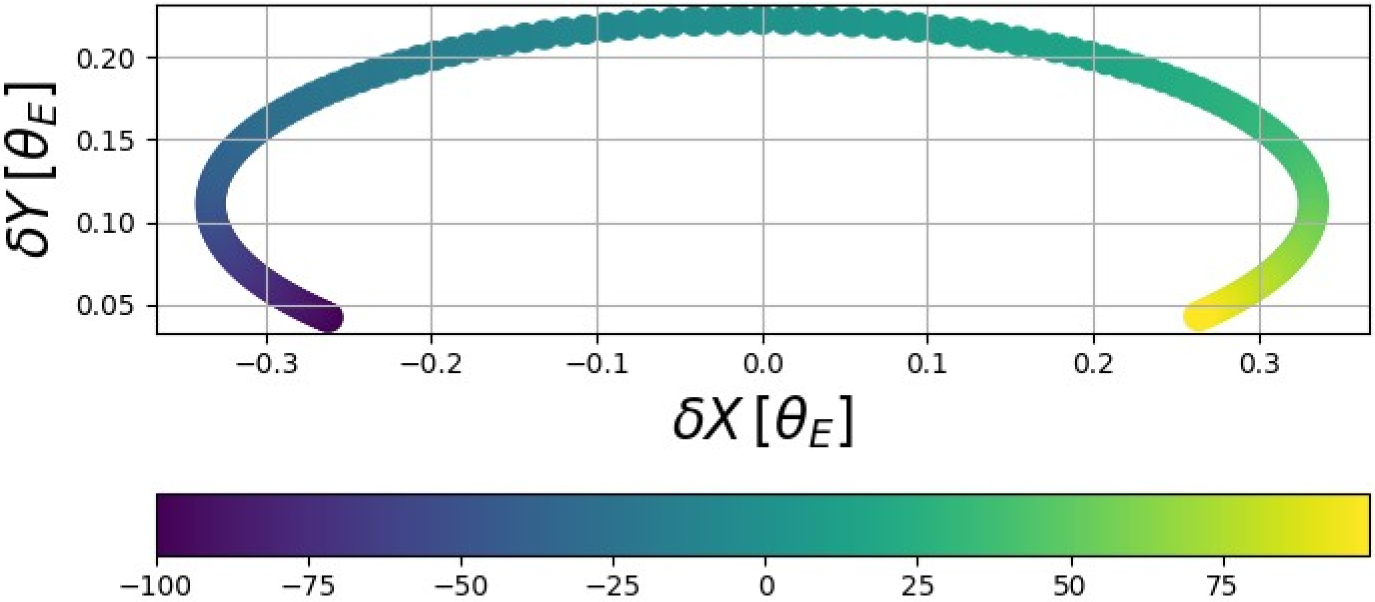
$$\delta(t)=\frac{\sum_{i=0}^{N_I} A_i z_i}{\sum_{i=0}^{N_I} A_i}-\zeta$$

N_L = 1 lens

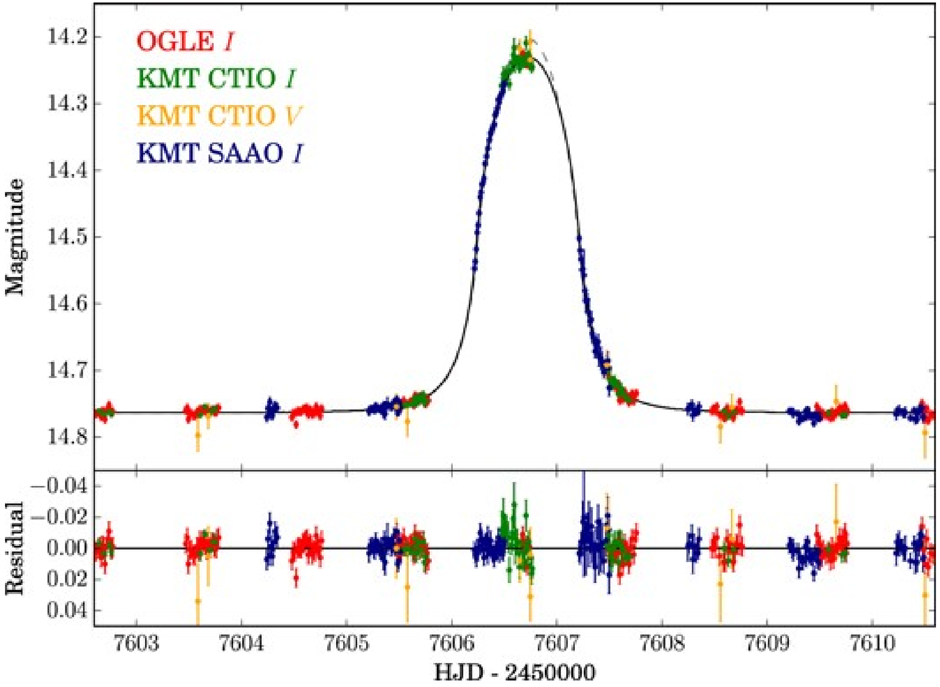
$$\boxed{\delta(t)=\frac{\zeta(t)}{\zeta(t)^2+2}}$$



Lens equation : 1 lens



Rogue planets



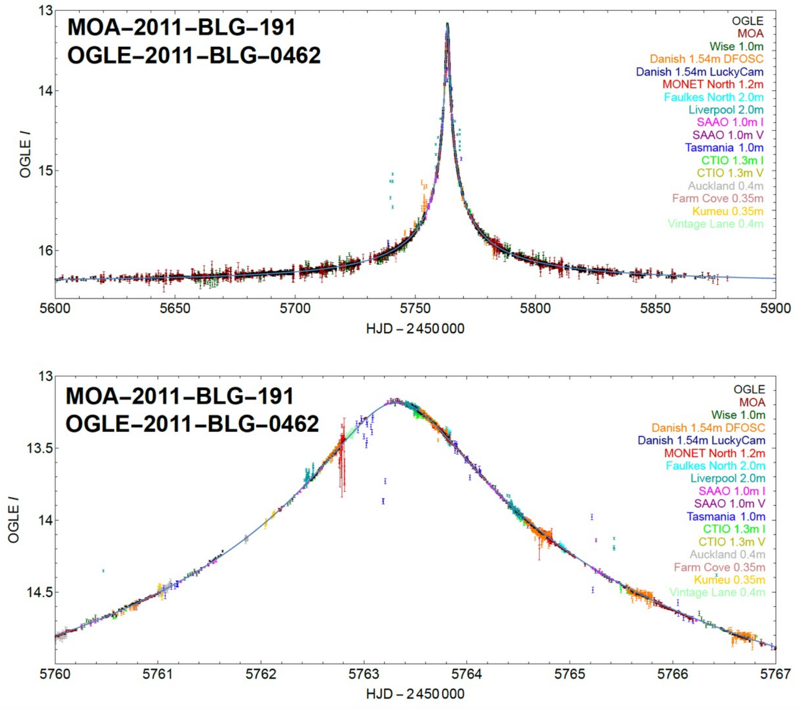
Mróz et al, 2018

Neptune-mass planet in the Disk

OR

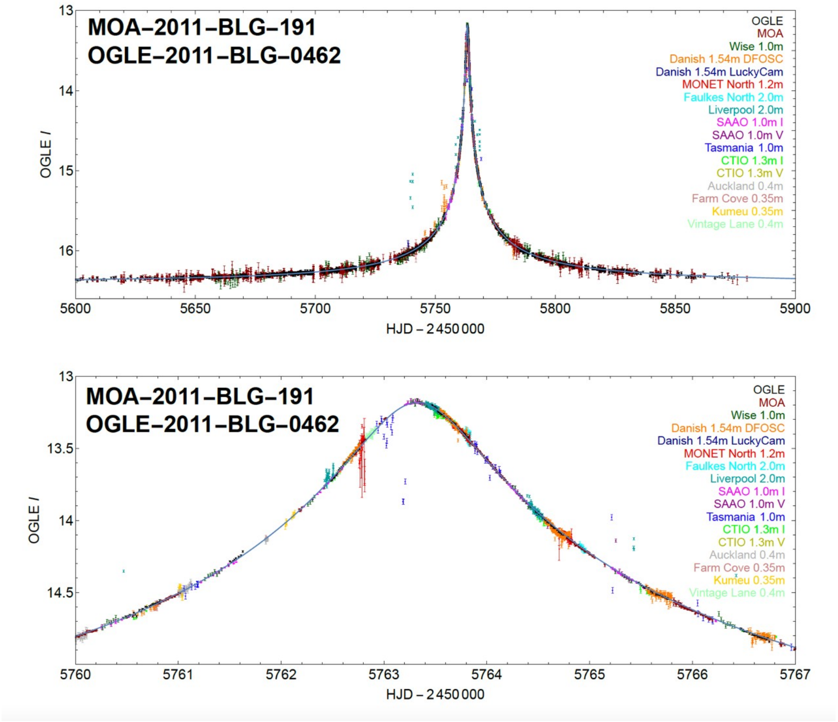
Saturn-mass planet in the Bulge

Isolated Stellar Mass Black Hole



Sahu et al. 2022
Lam et al. 2022
Mróz et al, 2018

Isolated Stellar Mass Black Hole



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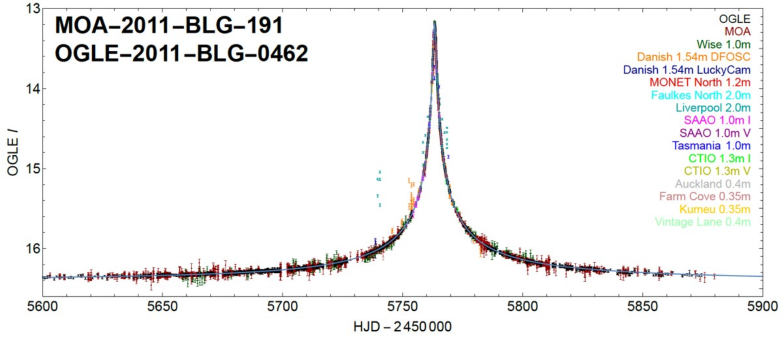
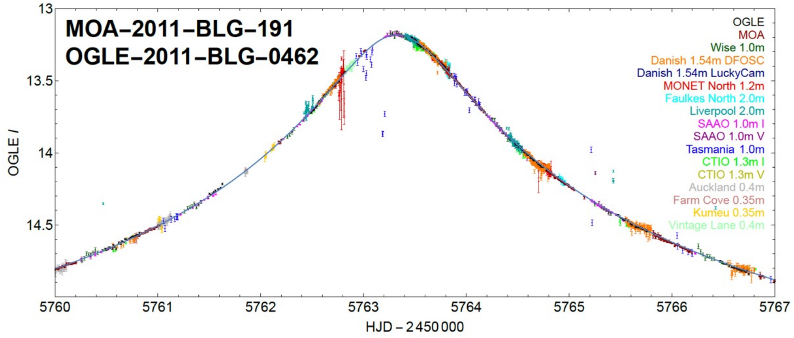
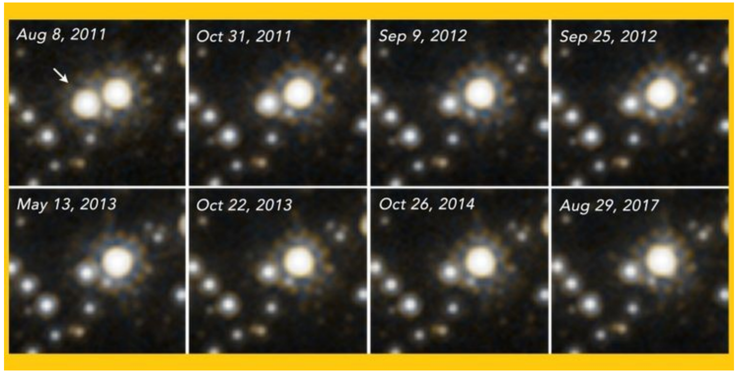
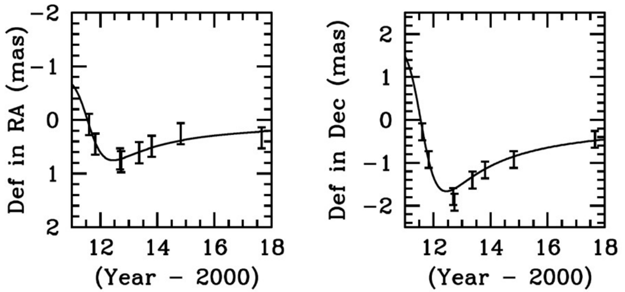
$$\pi_E = \frac{\pi_{rel}}{\theta_E}$$ (see V. Bozza talk)

Isolated Stellar Mass Black Hole









Sahu et al. 2022
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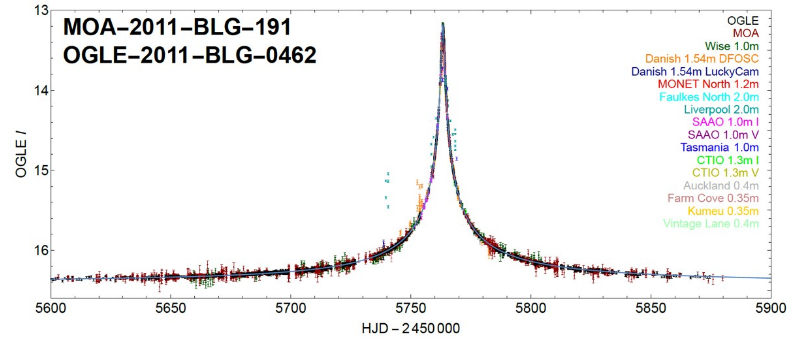
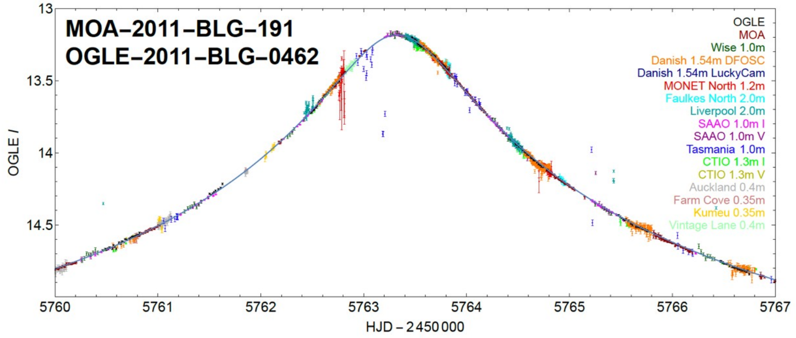
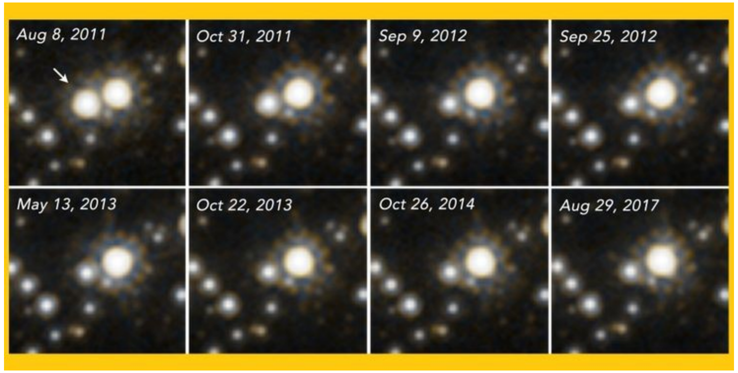
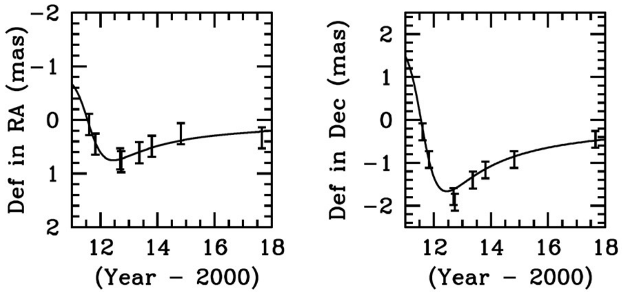
$$\pi_E = \frac{\pi_{rel}}{\theta_E}$$ (see V. Bozza talk)

Isolated Stellar Mass Black Hole







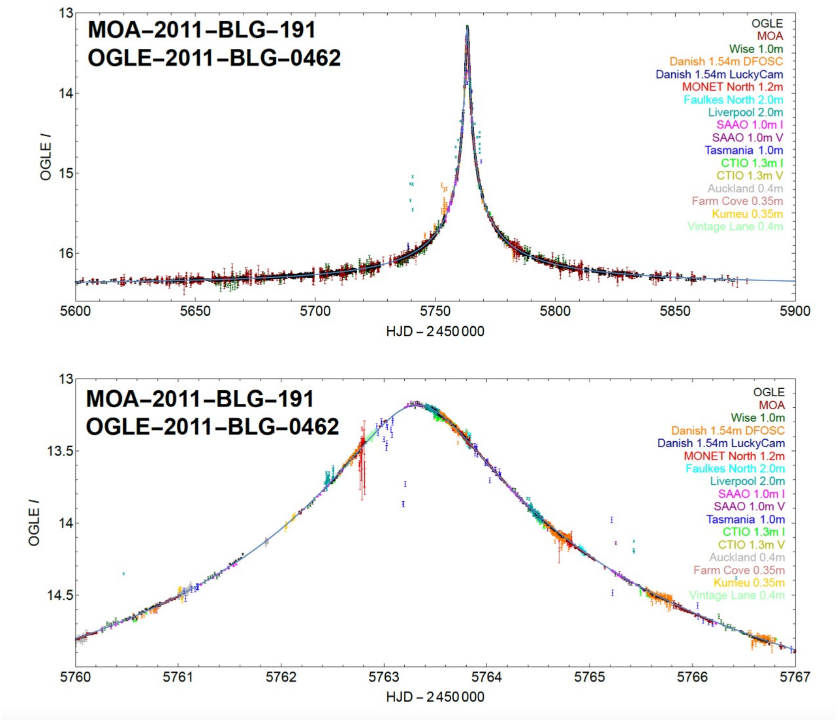


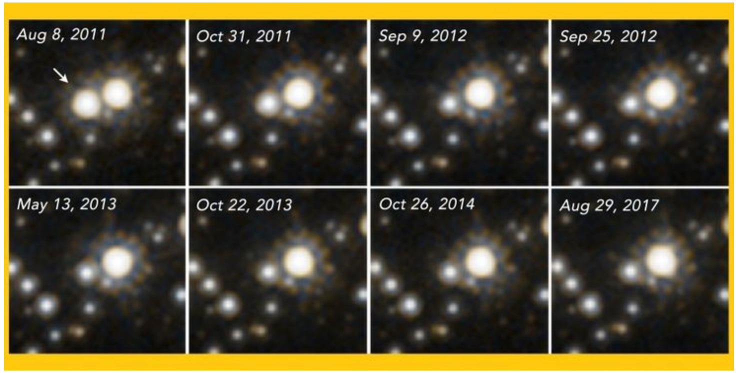
θ_E

Sahu et al. 2022
Lam et al. 2022
Mróz et al, 2018

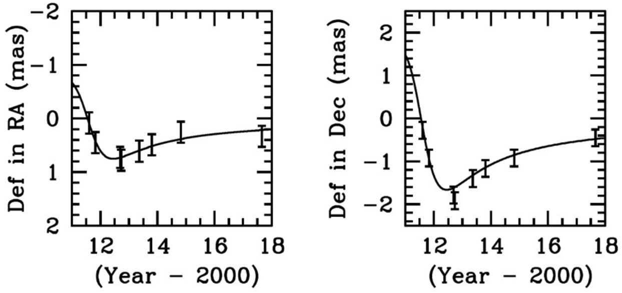
$$\pi_E = \frac{\pi_{rel}}{\theta_E}$$ (see V. Bozza talk)

Isolated Stellar Mass Black Hole





$$M_L = \frac{\theta_E}{\kappa \pi_E}$$



$$\pi_E = \frac{\pi_{rel}}{\theta_E}$$ (see V. Bozza talk)

$$\theta_E$$

Sahu et al. 2022
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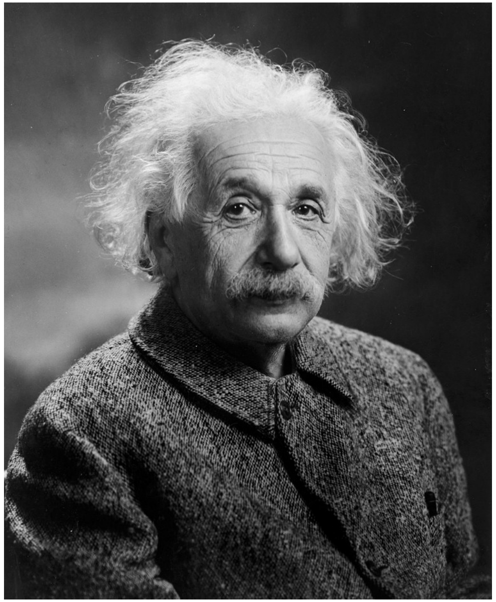
Optical depth/Event rate



Albert Einstein
(1879-1955)

“Of course, there is no hope of observing this phenomenon directly”

Einstein, Science, 1936

Optical depth/Event rate

Optical depth τ : probability of a given star to be microlensed

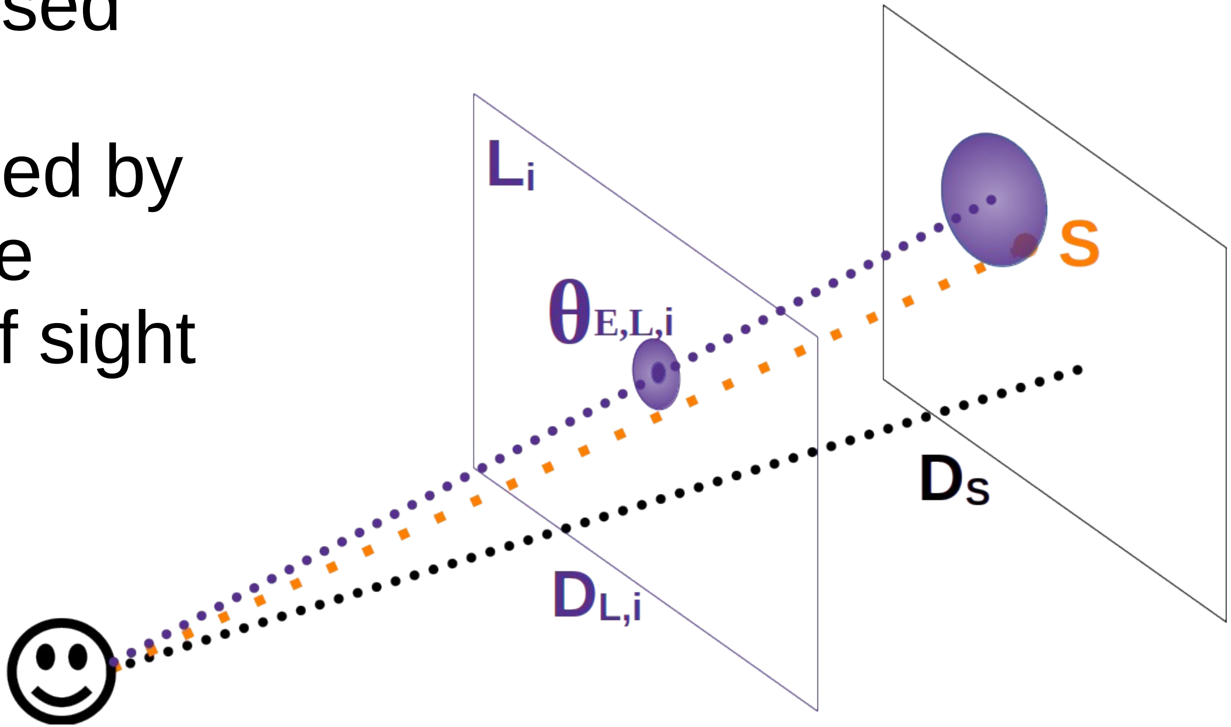
(see F. Zohrabi talk)

Optical depth/Event rate

Optical depth τ : probability of a given star to be microlensed

(see F. Zohrabi talk)

Fraction of the sky covered by all Einstein rings of all the potential along the line of sight (and with $|\zeta_0|<1$)



$$d\tau = n(D_L)\pi R_E^2 dD_L$$

Optical depth/Event rate

$$d\tau = \frac{\rho\left(D_L\right)}{M_L}\pi\frac{4GM_L}{c^2}\frac{D_L\left(D_S - D_L\right)}{D_S}dD_L$$

Optical depth/Event rate

$$d\tau = \frac{\rho(D_L)}{M_L} \pi \frac{4GM_L}{c^2} \frac{D_L(D_S - D_L)}{D_S} dD_L$$

$$\boxed{\tau = \int_0^{D_s} d\tau = \frac{4\pi G}{c^2} \int_0^{D_s} \rho(D_l) \frac{D_l(D_s - D_l)}{D_s} dD_l}$$

Towards the Bulge

$$\rho_0 \approx 0.1\, M_{\odot} \cdot pc^{-3}, D_S = 8\, kpc, \tau \approx 10^{-6}$$

Optical depth/Event rate

Event rate Γ :   number of new microlensing events per source per unit of time

Optical depth/Event rate

Event rate Γ : number of new microlensing events per source per unit of time

Assuming a standard transverse veolicity $v_\perp$ (and with $|\zeta_0|<1$)

$$\boxed{\Gamma = \int_0^{D_s} d\Gamma = \int_0^{D_S} 2 R_E v_\perp(D_L) n(D_L) dD_L}$$

$$\Gamma \approx \frac{2\tau}{\pi \overline{t_E}}$$

Towards the Bulge

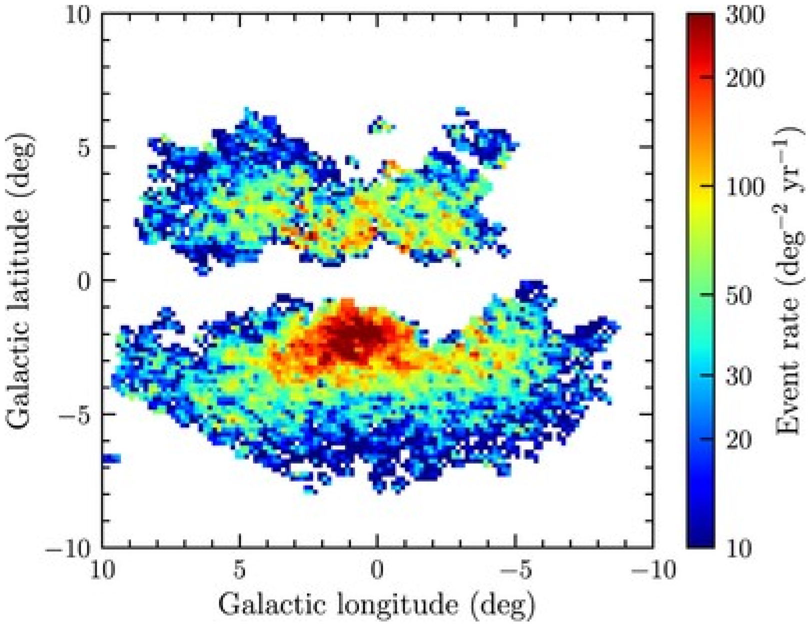
$$\overline{t_E} \approx 20\, days$$

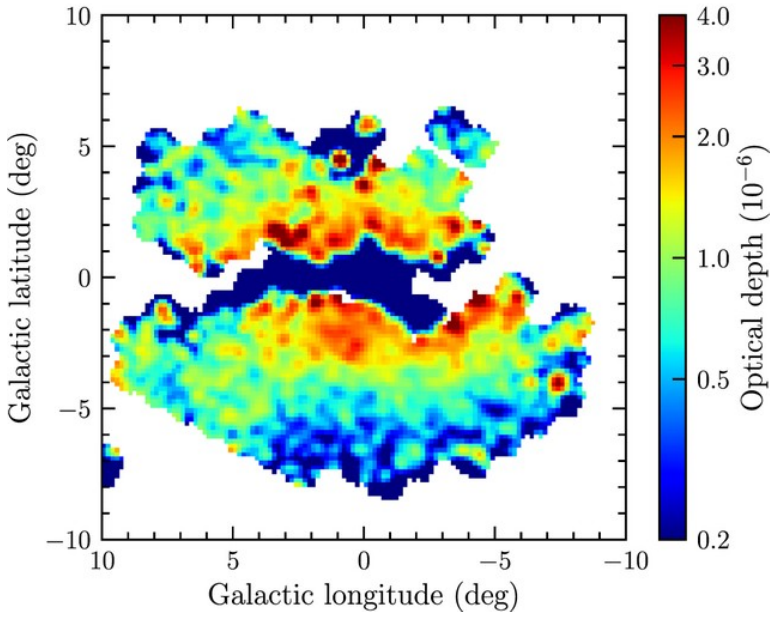
$$N_S = 200 \times 10^6$$

$$\Delta t \approx 180\, days$$

$$N_{events} \approx 1000\, year^{-1}$$

Optical depth/Event rate





Mróz et al, 2019

Lens equation : 2 lenses
(aka where the fun begins)

Lens equation : 2 lenses

$$\zeta = z - \frac{\epsilon_1}{\bar{z} - \overline{z_1}} - \frac{\epsilon_2}{\bar{z} - \overline{z_2}}$$

$$q = \frac{\epsilon_2}{\epsilon_1} \qquad s = z_2 - z_1$$

Polynomial of order 5

Lens equation : 2 lenses

$$\zeta = z - \frac{\epsilon_1}{\overline{z} - \overline{z_1}} - \frac{\epsilon_2}{\overline{z} - \overline{z_2}}$$

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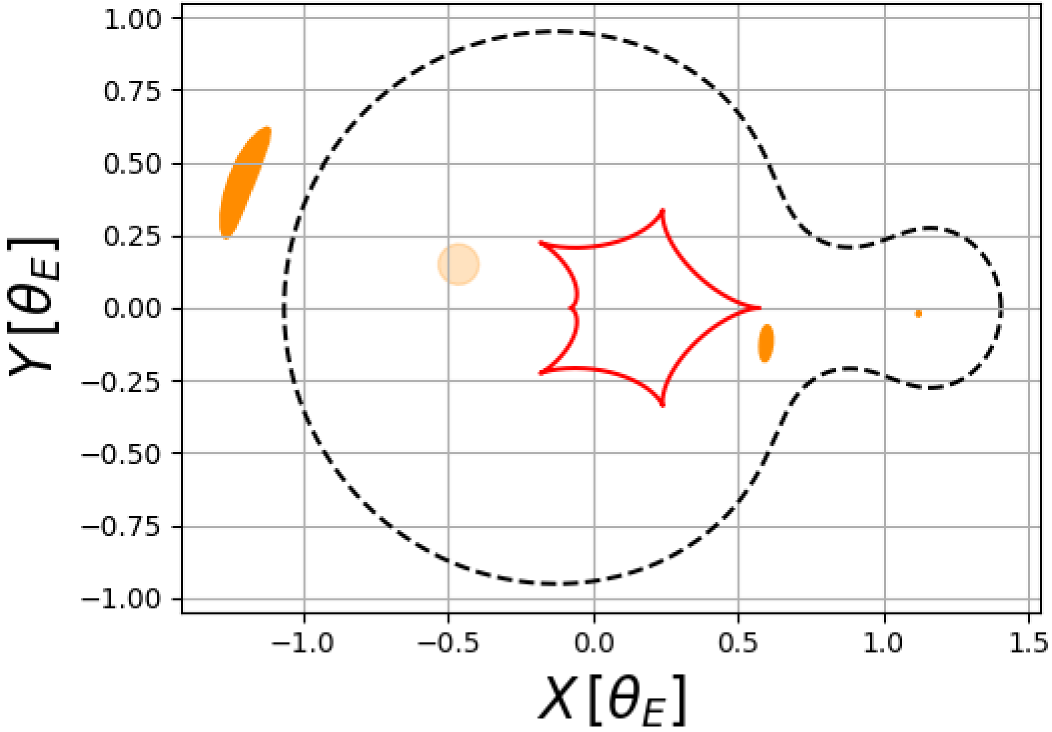


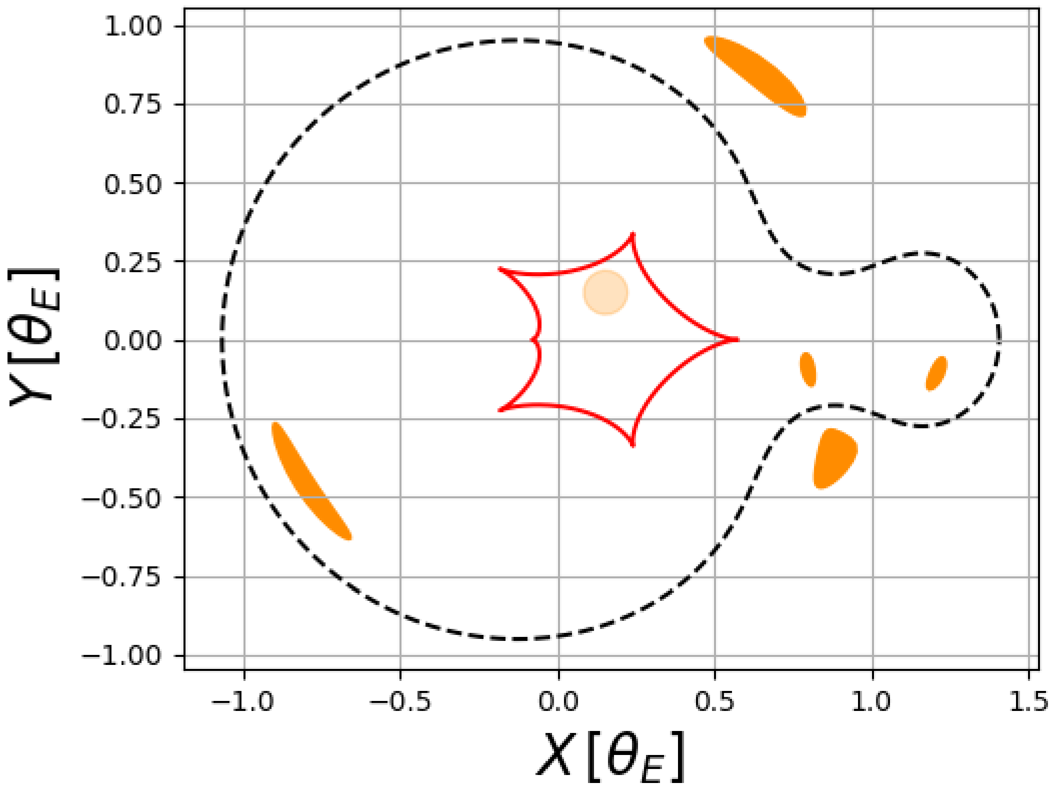
Polynomial of order 5

2 « ghost » images

Lens equation : 2 lenses

$$\zeta = z - \frac{\epsilon_1}{\bar{z} - \bar{z}_1} - \frac{\epsilon_2}{\bar{z} - \bar{z}_2}$$

$$q = \frac{\epsilon_2}{\epsilon_1} \qquad s = z_2 - z_1$$



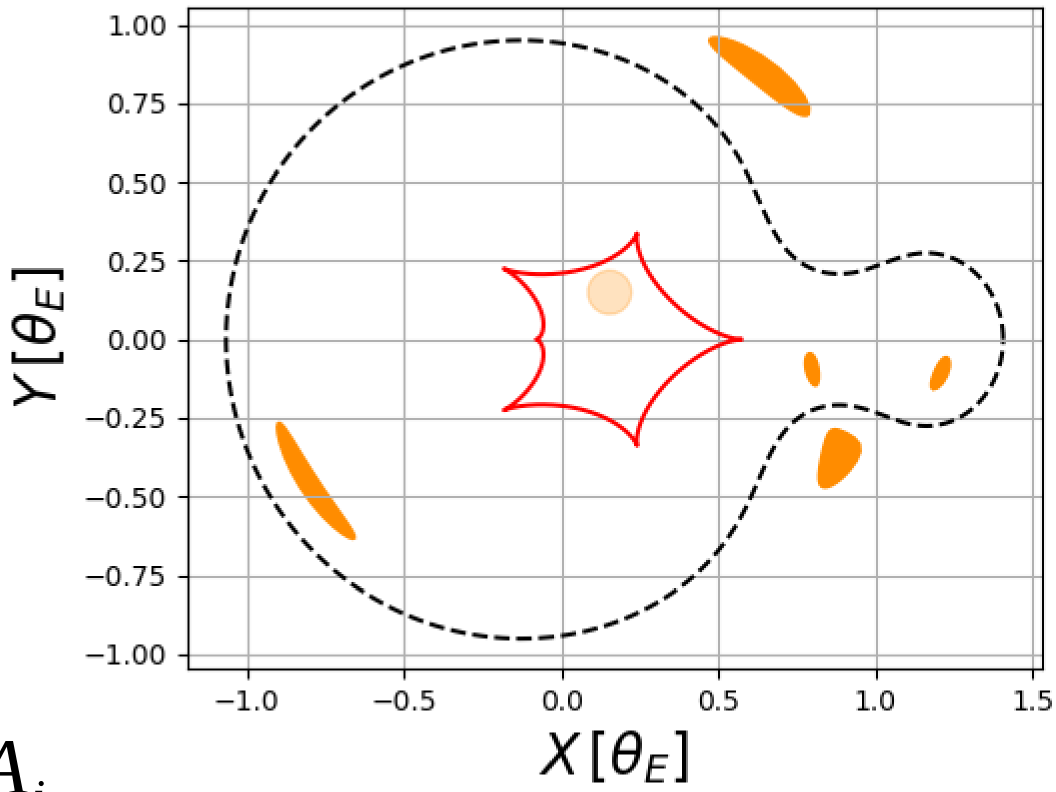
+2 real images (odd number theorem)

Lens equation : 2 lenses

More generally, the magnification of an image is given by the inverse of the Jacobian of the mapping

$$\boxed{A_i = \frac{area_i}{area_S} = \frac{1}{|det\, J|}}$$

$$det\, J = 1 - \frac{\partial \zeta}{\partial \bar{z}} \frac{\partial \bar{\zeta}}{\partial \bar{z}} \qquad A = \sum_{i}^{N_I} A_i$$



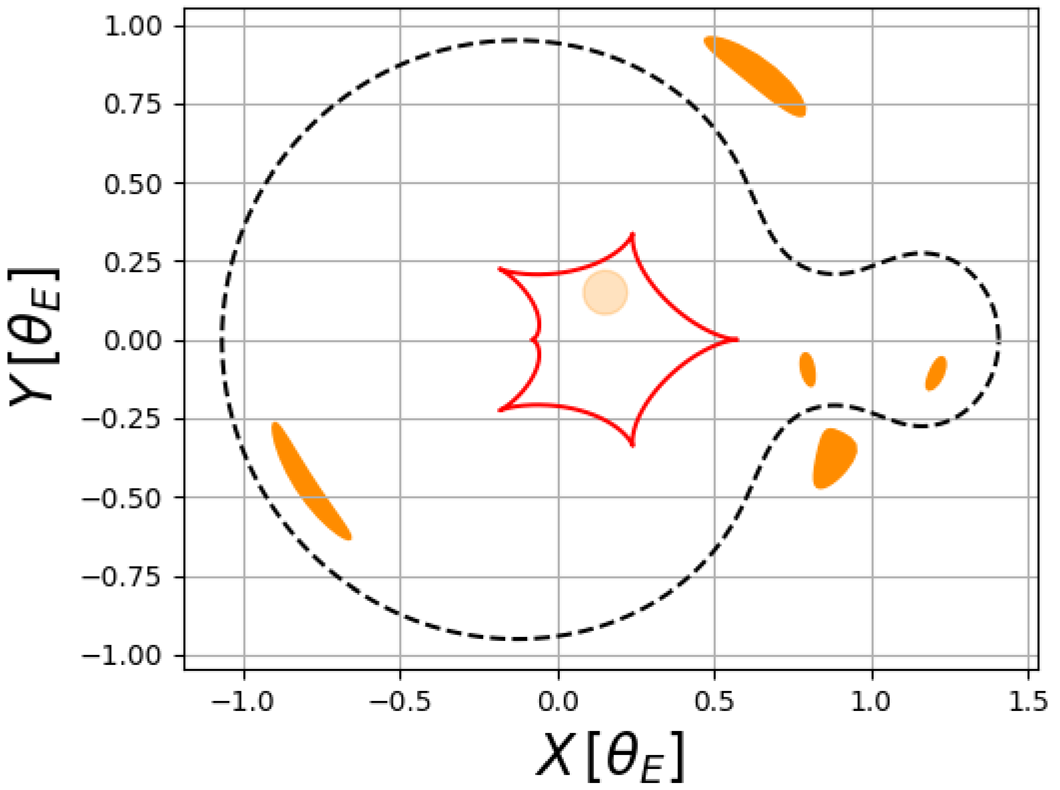
Lens equation : 2 lenses

Mapping is singular when det J = 0

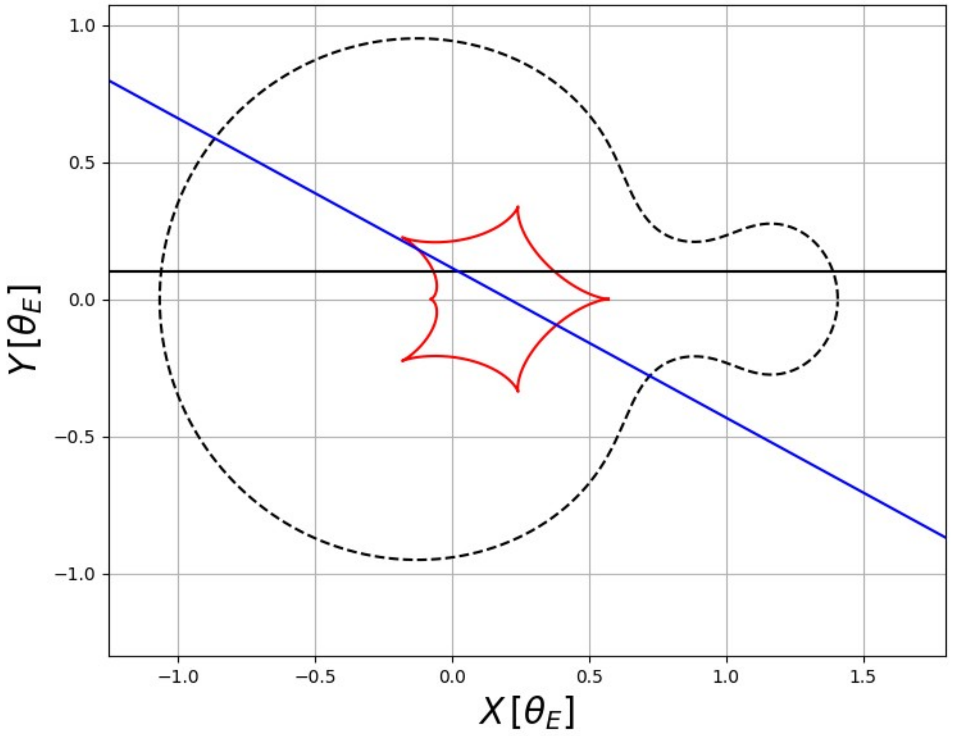
Corresponding image positions are called **critical curves**

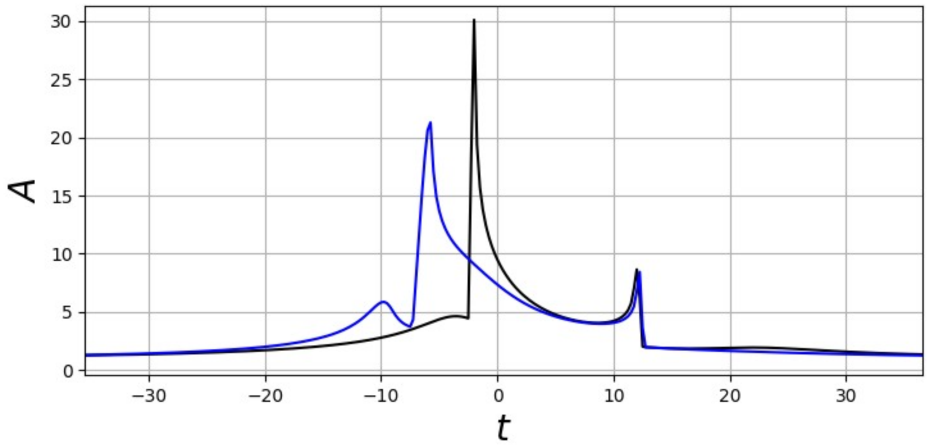
Corresponding source positions are called **caustics**

Inside caustics $A \geq 3$



Lens equation : 2 lenses



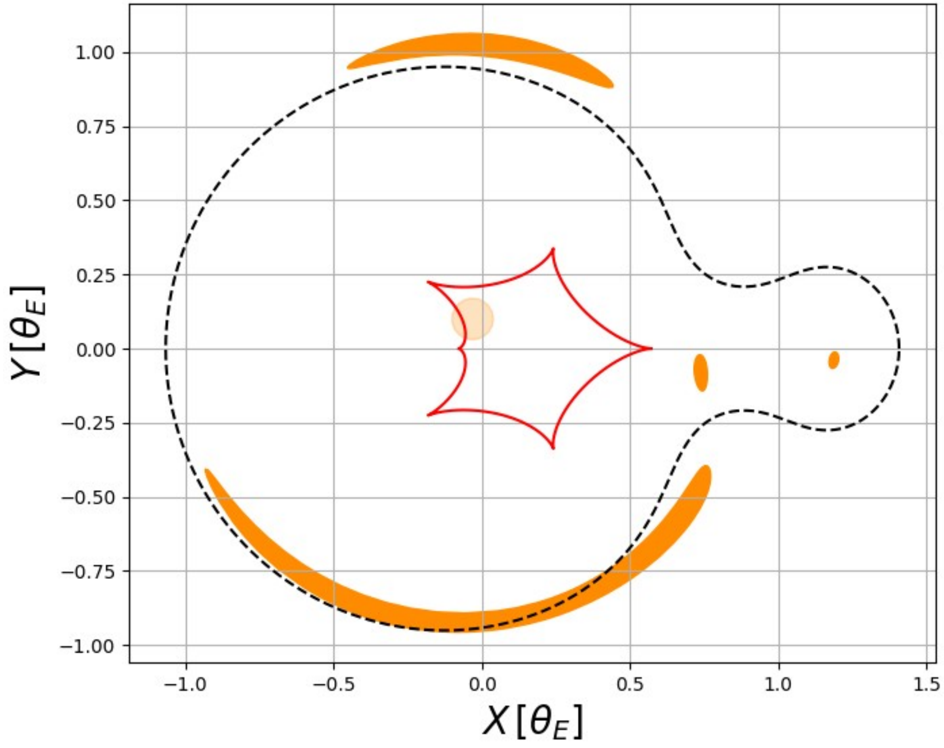


Lens equation : 2 lenses

Because sources are not points-like, an additional difficulty occurs at caustic crossing

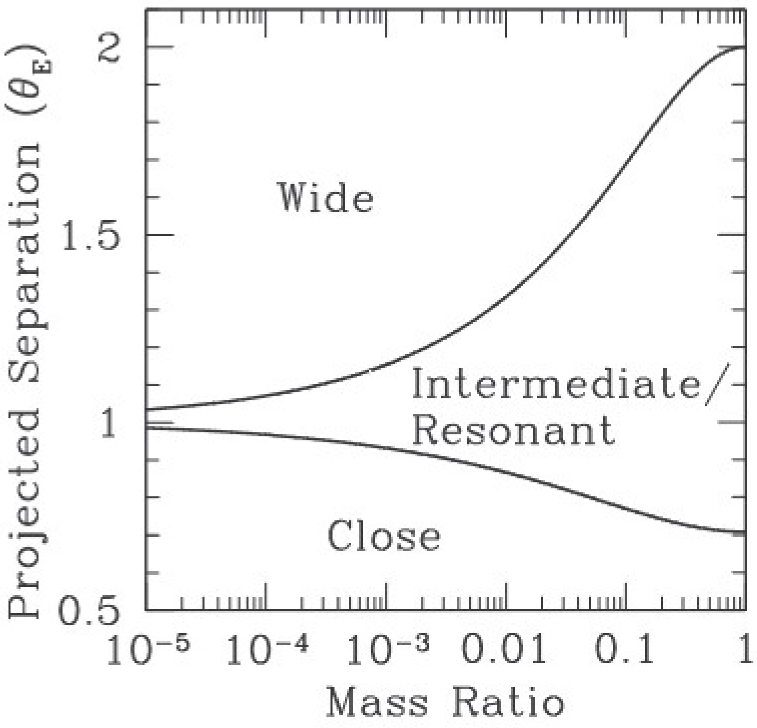
- contour integration technics

- ray shooting technics

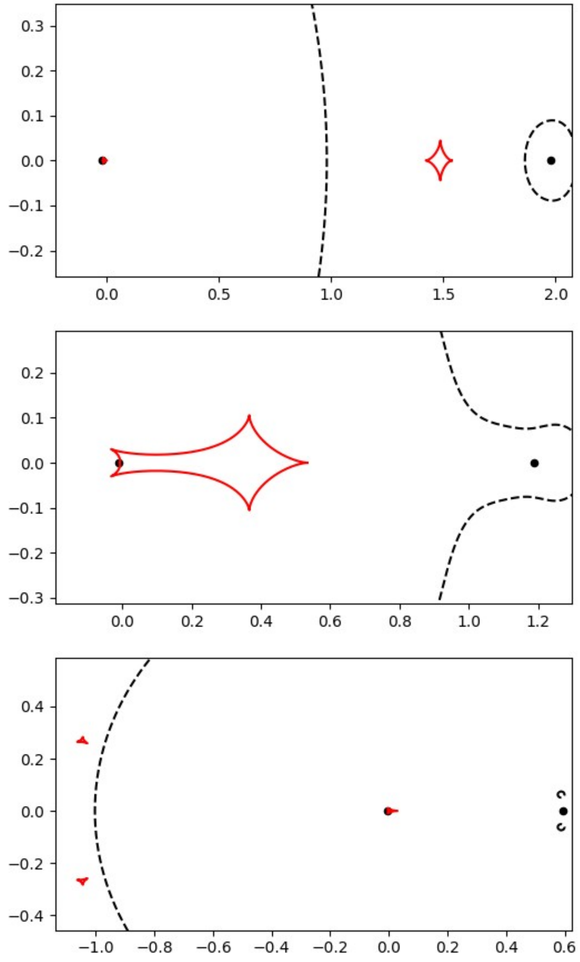


(see V. Bozza and S. Ishida talks)

Lens equation : 2 lenses



Gaudi B.S. 2010

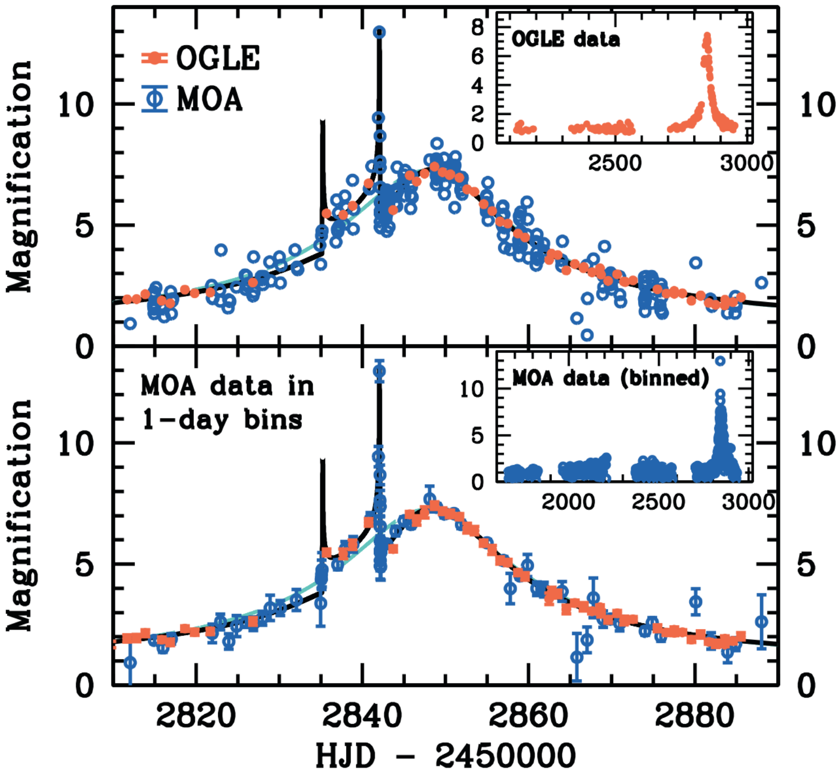


s = 2
q = 0.01

s = 1.2
q = 0.01

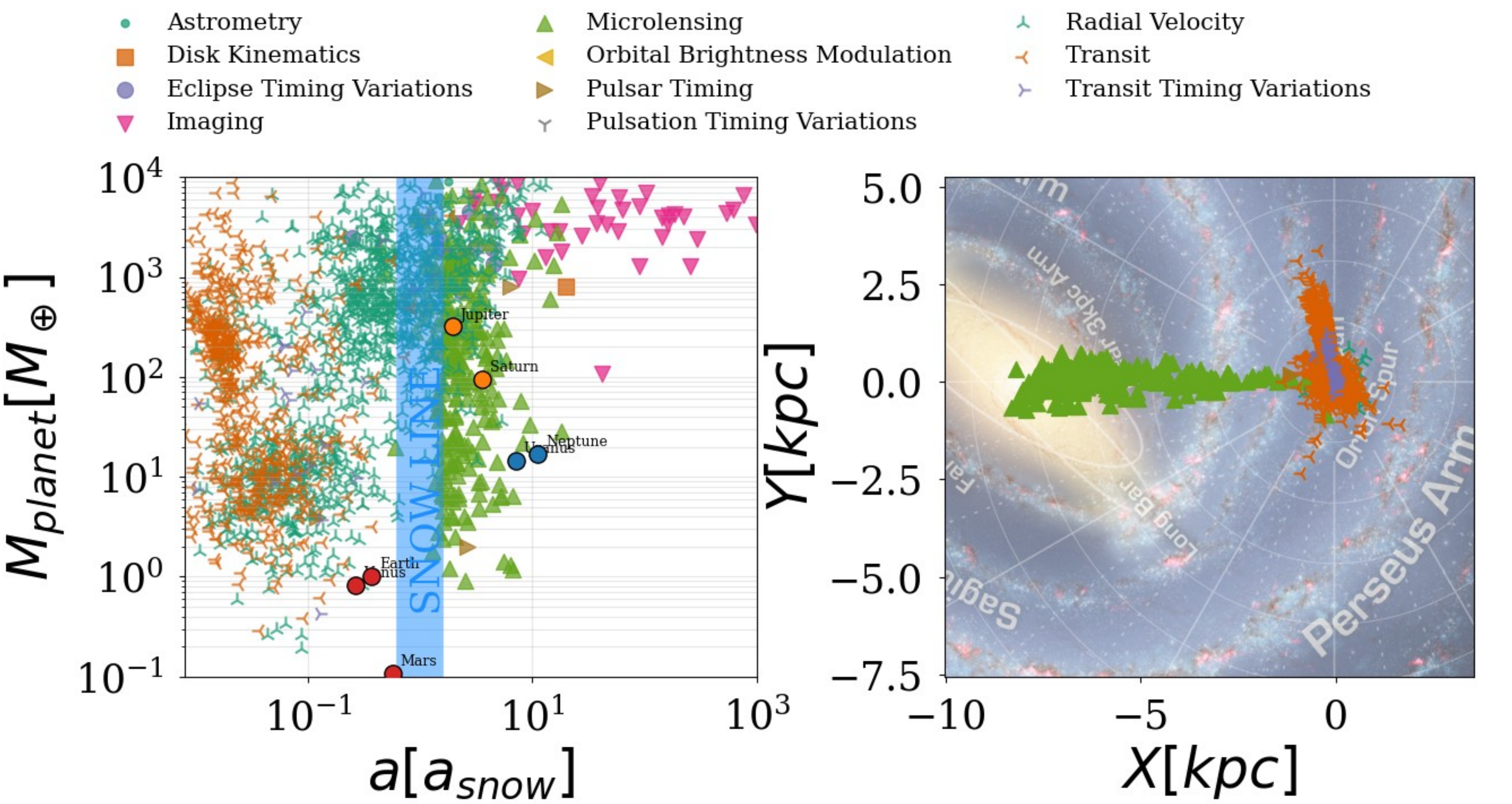
s = 0.6
q = 0.01

OGLE-2003-BLG-235/MOA-2003-BLG-53: the first microlensing planet



Bond et al., ApJ, 2004

Microlensing is complementary to other methods



NASA Exoplanet Science Archive (21 June 2026)

Lens equation : 3 lenses and more

Lens equation : 3 lenses and more

Sorry, not enough time !

To go further

General about (micro)lensing

Peter Schneider, Jürgen Ehlers, Emilio E. Falco
Gould A. 2000
Gaudi B.S 2010
Tsapras Y. 2018

Lens Equation/Caustics

Witt H.J. 1990
Witt & Mao 1995

Astrometric microlensing

Dominik & Sahu 2000
Nucita et al. 2017
Sahu et al. 2017
McGill et al. 2022

Interferometry

Cassan & Ranc 2016
Dong et al. 2019

Optical depth/Event rate

Paczyński B. 1986
Tisserand et al. 2007
Wyrzykowski et al. 2011
Mróz et al. 2019

Codes

https://github.com/ebachelet/pyLIMA
https://github.com/rpoleski/MulensModel
https://github.com/valboz/VBMicrolensing
https://github.com/valboz/RTModel
https://github.com/MovingUniverseLab/BAGLE_Microlensing